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ABSTRACT

This is one in a series of manuals for teachers using SMSG high school supplementary materials. The pamphlet includes commentaries on the sections of the student's booklet, answers to the exercises, and sample test questions. Topics covered include the coordinate system, distance formula, planes and first degree equations in three variables, the graph of a first degree equation in three variables, intersecting planes, and parametric equations. (MP)

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SCHOOL
MATHEMATICS
STUDY GROUP

SP-22

SUPPLEMENTARY and ENRICHMENT SERIES

SYSTEMS OF FIRST DEGREE EQUATIONS IN THREE VARIABLES

Teachers' Commentary

Edited by Jean M. Calloway

Financial support for the School Mathematics Study Group has been provided by the National Science Foundation.

PREFACE

Mathematics is such a vast and rapidly expanding field of study that there are inevitably many important and fascinating aspects of the subject which, though within the grasp of secondary school students, do not find a place in the curriculum simply because of a lack of time.

Many classes and individual students, however, may find time to pursue mathematical topics of special interest to them. This series of pamphlets, whose production is sponsored by the School Mathematics Study Group, is designed to make material for such study readily accessible in classroom quantity.

Some of the pamphlets deal with material found in the regular curriculum but in a more extensive or intensive manner or from a novel point of view. Others deal with topics not usually found at all in the standard curriculum. It is hoped that these pamphlets will find use in classrooms in at least two ways. Some of the pamphlets produced could be used to extend the work done by a class with a regular textbook but others could be used profitably when teachers want to experiment with a treatment of a topic different from the treatment in the regular text of the class. In all cases, the pamphlets are designed to promote the enjoyment of studying mathematics.

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CONTENTS

TEACHERS COMMENTARY AND ANSWERS

SYSTEMS OF FIRST DEGREE EQUATIONS IN THREE VARIABLES

0 Introduction

Systems of first degree equations arise in many branches of mathematics and science as well as in many modern theories of economics and in business problems, particularly those concerned with inventory and production questions. The geometrical significance of the subject is emphasized by our presentation of the geometry along with the algebra of the problem.

With the current use of computing machines to solve engineering and scientific problems, this subject has become one of the most important branches of applied mathematics. Every day industrial and research organizations must solve systems of first degree equations, some of them with hundreds of equations and hundreds of variables. A thorough understanding of the simpler cases is therefore a necessity for any one hoping to take almost any kind of mathematical job in industry or scientific research.

The central problem studied in this chapter is an algebraic one: under what circumstances do two or more equations in three variables with real coefficients have common solutions, and if there are common solutions, how many are there and how are they related to one another? Because we restrict our attention to first degree equations with real coefficients having only three variables, we are able to translate the problem into geometric language. This translation makes it possible to cast our results in the form of statements about planes in three dimensional space in such a way that statements about common solutions of the equations become statements about configurations of planes and their intersections. The insights gained in this way are perhaps most strikingly illustrated by the diagrams in Figure 9b where the many types of intersection and parallelism of planes are used to describe the types of solution sets that may be expected when a system of three first degree equations in three variables is studied. In this presentation, Section 9 gives cases where there are solutions, and also gives cases where the solution set is empty. Although it is not essential that the student understand all these cases, some students will enjoy the opportunity to see a classification of this kind. Thus, those students who learn to handle the ideas presented in discussing the correspondence between the geometry and algebra of these (and other) systems of equations will benefit from the ability to visualize the geometry. If they go on to the study of mathematics at college, they will find that the development of this ability gives them real advantages.

Degrees of Freedom

One of the basic ideas that will throw light on the approach to the problems studied in this chapter is the concept of degrees of freedom. An understanding of this concept will improve the teacher's intuition about all the problems discussed here. (For a more sophisticated treatment of this question and of linear dependence, see Birkhoff-MacLane, Survey of Modern Algebra, p. 166ff.)

A point in space has 3 coordinates (x,y,z). It is said to have 3 degrees of freedom since each of the variables may be assigned arbitrary values. As each of x, y, and z assumes all possible real values, the point (x,y,z) assumes all possible positions in space. If, however, the values of the variables are constrained to satisfy a single equation (here an equation of first degree - the equation of a plane), the number of variables that may be assigned arbitrary values is reduced to 2. The point now has only 2 degrees of freedom and is constrained to remain in the plane whose equation is given. We say that the number triples in the solution set of the equation can be described in terms of 2 parameters. (This case is treated in the first part of Section 4, but the word parameter is not used in the text.)

If 2 first degree equations are given there are 3 possibilities:

1. In the most interesting case there are points (x,y,z) whose coordinates satisfy the 2 equations; the 2 equations impose 2 independent conditions on the point, and only one variable may be assigned arbitrarily. The point (x,y,z) now has only one degree of freedom. The number triples in the solution set of the system of 2 equations can be described in terms of a single parameter. The point is constrained to remain on a line--the line of intersection of the 2 given planes.

   Section 8 develops methods for describing the line of intersection in different ways, depending on which variable is assigned arbitrarily (which variable serves as parameter).

2. In the second case, the 2 equations are inconsistent. They represent parallel planes (discussed below and in starred Section 7), and no number triple can satisfy both equations. Here there is no point that is common to both planes.

3. In the third case the 2 equations represent the same plane, and we have actually only 1 condition on the coordinates of points in this plane. Again the number triples in the solution set are described in terms of 2 parameters. (This case is also discussed below and in Section 7.)

If a third first degree equation, consistent with the first two and independent of them, is given, an additional condition is imposed on the number triple (x,y,z). In this case, no variable may be chosen arbitrarily. The coordinates (x,y,z) are completely determined, and the point (x,y,z) is the single point of intersection of the three planes. This is one of the cases studied in Section 9. The cases in which the systems are dependent reduce to one of the two cases studied above: a line of intersection (one degree of freedom)--one parameter, or a plane of intersection (two degrees of freedom)--two parameters. If the system is inconsistent (and this can happen in a variety of ways, as illustrated in Figure 9b), then there is again no point that is common to the three planes.

The manipulations that enable us to find the solution set for a system of equations are justified by the fact that the given system is consistently replaced by an equivalent system. The new system is equivalent to the old because the new equations are derived by taking linear combinations of the expressions defining the given equa ions. Hence the planes defined by the new equations pass through the intersection of the given planes; when we have described the solution set of the new equations, we have also described the solution set of the given equations.

General Comments: An Outline of our Procedure--Suggestions to the Teacher.

We give here an outline of our procedure in this pamphlet and an indication, in certain parts, of teaching techniques that may make the presentation easier or of aspects of the problem that are not developed in the pamphlet but may be useful for the teacher to know.

The Purpose of the First Three Sections (Sections 1, 2, 3).

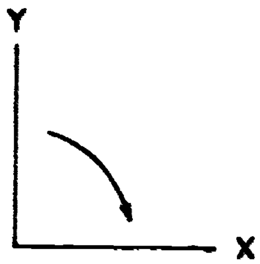
The first three sections are included in order to establish our basic geometric-algebraic correspondence; namely, the theorem that the equation of a plane is always of first degree, and that a first degree equation always represents a plane. There are several points that should be made here:

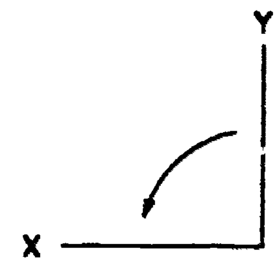
1. Comments. The Coordinate System (Section 1).

The coordinate system used is a "right-handed" one. This is an arbitrary choice that is made because of the widespread use of this system in physics, vector analysis, and other courses in mathematics and its applications.

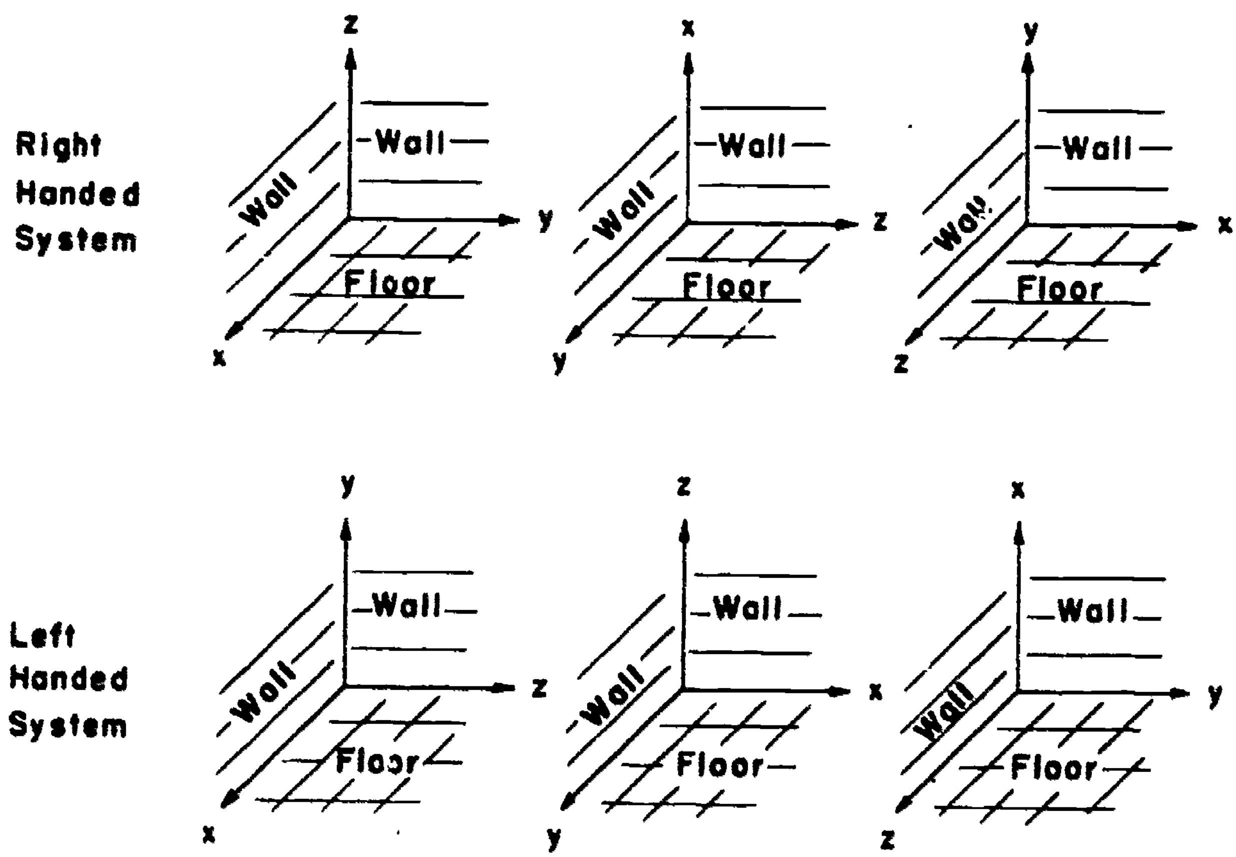
What do we mean by a "right-hand" or a "left-handed" system?

The difference between these systems m•y be expressed somewhat picturesquely perhaps, as follows: In a right-handed coordinate system, a person impaled on the positive Z-axis and looking toward the XY-plane views it just as he always did in plane geometry--the positive X-axis positive to the <u>right</u> of the positive Y-axis. In a left-handed system our observer on the positive Z-axis, on looking toward the XY-plane sees the positive X-axis extending to the <u>left</u> of the positive Y-axis.



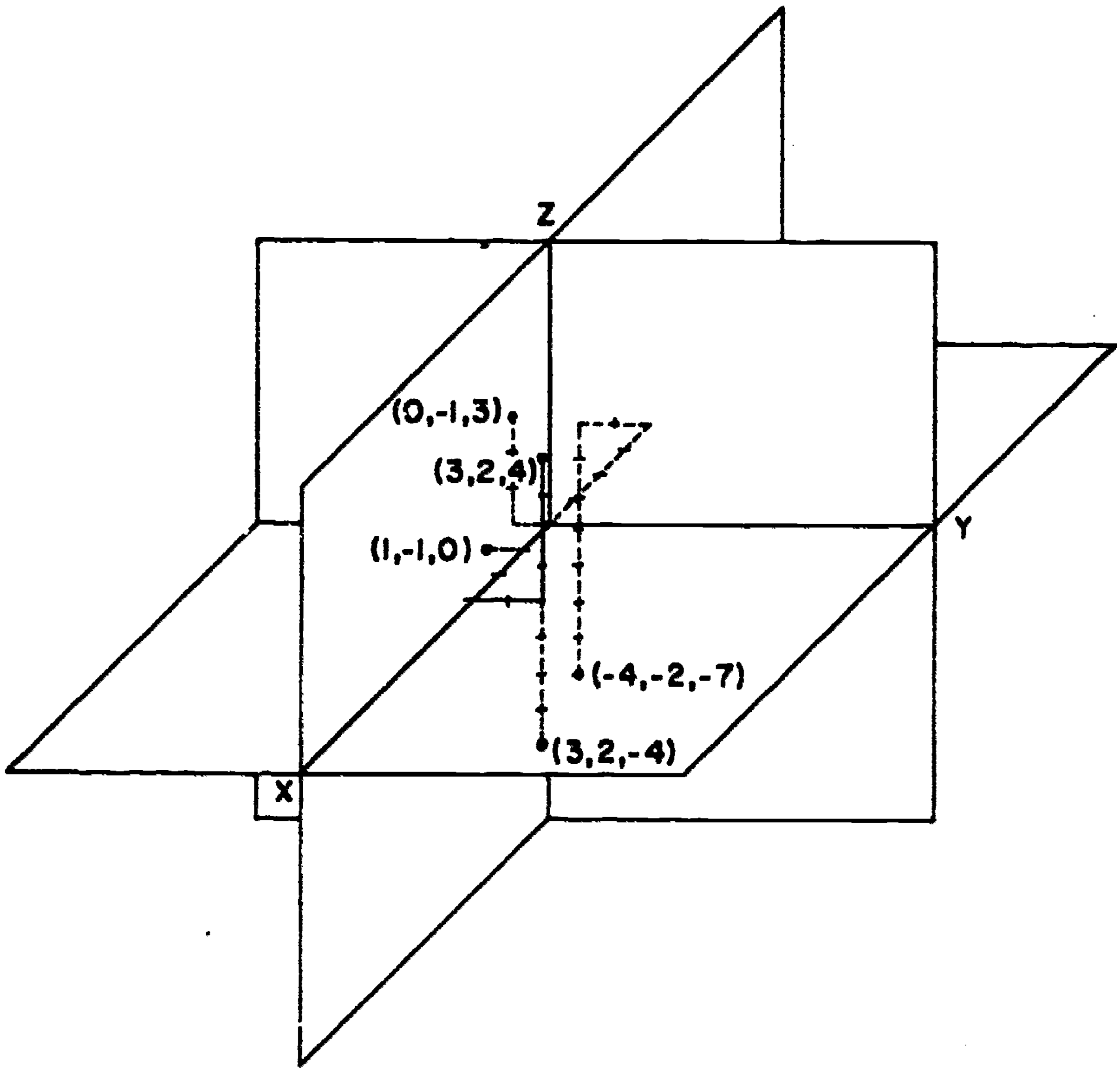


The following sketches illustrate a variety of the views an observer may have of each type of system.



Exercises 1. - Answers.

1. See graph.
2. Point not plotted.
3. See graph.
4. Point not plotted.
5. See graph.
6. Point not plotted.
7. See graph.
8. Point not plotted.
9. See graph.
10. Point not plotted.



11. (a) On the yz-plane

(b) On a plane || to the yz-plane cutting the x-axis at 2.

(c) On a plane || to the yz-plane cutting the x-axis at -3.

12. (a) On the xz-plane

(b) On a plane || to the xz-plane and cutting the y-axis at 3.

13. (a) On a plane || to the xy-plane and cutting the z-axis at 2.

(b) On a plane || to the xy-plane and cutting the x-axis at -2.

14. A plane || to the z-axis and cutting the x and y-axis at 4.

---

2. Comments. The Distance Formula in Space.

In teaching the distance formula in space, many teachers have found that a box or other model constructed with pieces of hardware-cloth, window screen, or ordinary cardboard is very helpful. Even the corner of a room can be used to assist the student to visualize this, and other parts of geometry in space. The industrial arts teacher can be very helpful in providing large drawings for display purposes; and it may be possible to secure film strips that will show figures in three dimensions.

Exercises 2. - Answers.

1.	$4\sqrt{2}$	6.	29
2.	13	7.	3
3.	7	8.	$\sqrt{129}$
4.	5	9.	$\sqrt{41}$
5.	12	10.	$\sqrt{14}$

---

3. Comments. The Correspondence Between Planes and First Degree Equations in Three Variables.

In proving the theorem establishing the correspondence between planes in space and first degree equations in 3 variables (Section 3), we did not have available the customary techniques of solid analytic geometry for deriving the equation of a plane. Thus, instead of viewing the plane as the locus of points on lines perpendicular to a given line through a point on that line, we have adopted a different definition. We have viewed the plane as the locus of points equidistant from two given points. This definition enables us to derive the equation of the plane with no analytic machinery beyond the distance formula. Since our definition embodies a property that characterizes a plane, the equation we derive represents precisely the plane with all the properties studied in geometry. In particular, a pair of distinct planes are either parallel or they intersect in a line.

If the teacher is pressed for time, it is suggested that the proof in Section 3 be omitted. The student should then accept without proof the theorem that every plane in three dimensions can be represented by an equation of the form

$$Ax + By + Cz + D = 0$$

where A, B, C, D, are real constants, and A, B, C, are not all zero; and the converse theorem, that every equation of this form represents a plane.

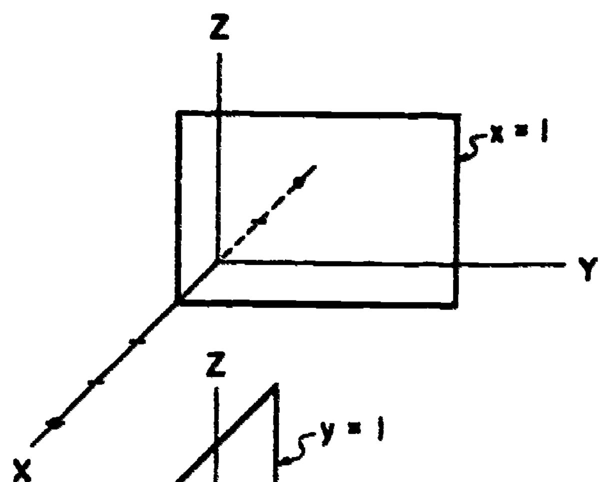
<u>Exercises</u> <u>3</u>. - <u>Answers</u>.

1. (a) $10x - 10y - 8z - 10 = 0$
   (b) $2x - 6y - 12z + 6 = 0$
   (c) $-20x + 4y - 8z = 0$
   (d) $4x + 4y - 16z = 32$
   (e) $6x + 8y - 6z = -14$
   (f) $4x - 8y + 12z = 0$

2. (a) $(4,0,0)$, $(-2,0,0)$.
   Plane has equation

   $$x = 1$$

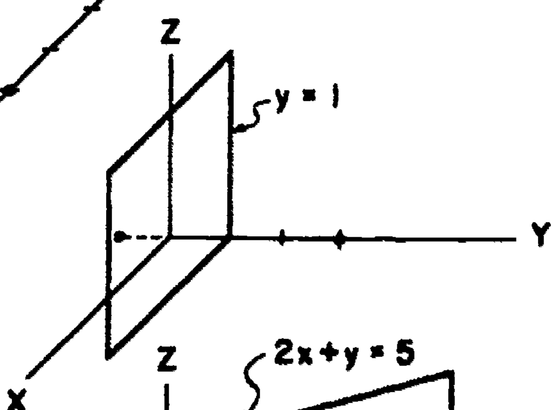
   Plane is parallel to YZ-plane and cuts the X-axis at 1 .



   (b) $(0,3,0)$, $(0,-1,0)$.
   Plane has equation

   $$y = 1$$

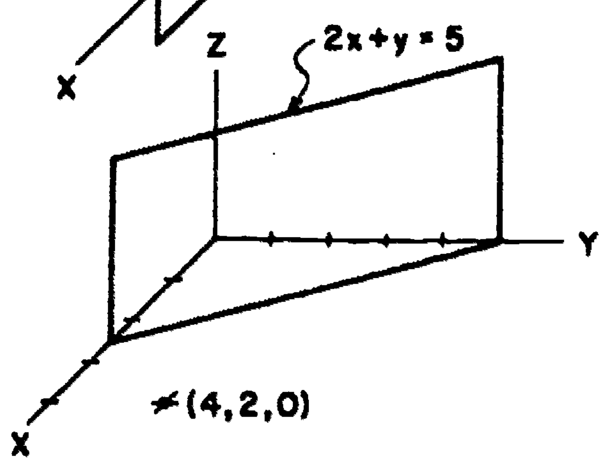
   Plane is parallel to XZ-plane and cuts the Y-axis at 1 .



   (c) $(0,0,0)$, $(4,2,0)$.
   Plane has equation

   $$8x + 4y = 20$$
   or $2x + y = 5$

   Plane is parallel to the Z-axis, and cuts the X-axis at $\frac{5}{2}$ and the Y-axis at 5 .



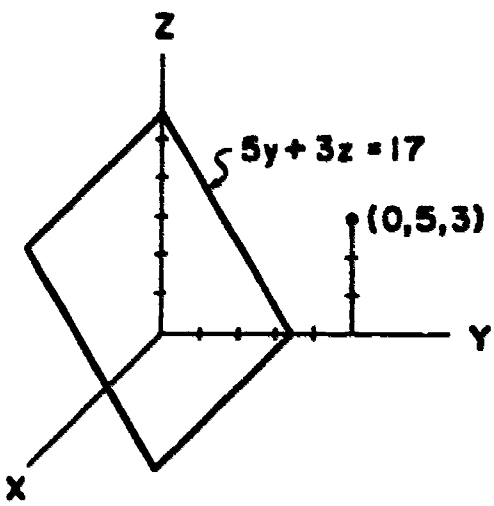
(d) (0,0,0) , (0,5,3).

Plane has equation

$$10y + 6z = 34$$

or $$5y + 3z = 17$$

Plane is parallel to the X-axis, and cuts the Y-axis at $\frac{17}{5}$ and the Z-axis at $\frac{17}{3}$.



3. Let $P(x,y,z)$ be any point on the plane belonging to the set of points equidistant from $R(a,b,c)$ and $S(-a,-b,-c)$. Since these points are symmetric with respect to the origin, we would expect the required plane to pass through the origin.

$$d(R,P) = d(S,P)$$

$$(x - a)^2 + (y - b)^2 + (z - c)^2 = (x + a)^2 + (y + b)^2 + (z + c)^2$$

$$-2ax - 2by - 2cz + a^2 + b^2 + c^2 = 2ax + 2by + 2cz + a^2 + b^2 + c^2$$

$$4ax + 4by + 4cz = 0$$

$$ax + by + cz = 0$$

Since (0,0,0) is a point in the solution set of this equation, the plane passes through the origin.

4, 5. Comments. The Graph of a First Degree Equation in Three Variables.

The correspondence between a plane and a first degree equation is introduced to throw light on the algebraic problem. The ability to draw the graph of an equation will enable the student to gain insight into some of the special situations that may occur when, in the later sections, we study the solution sets of systems of two or three equations. In Section 4 and 5 we try to develop this ability to draw graphs, first for the special planes that are parallel to an axis but not parallel to a coordinate plane (one variable has a zero coefficient, e.g., $x + y = 4$); second for planes parallel to a coordinate plane (two variables have zero coefficients--the equation gives a constant value for one coordinate, e.g., $x = 3$); and last for planes that have equations with no coefficients equal to zero. In all these cases, we consider the trace of the plane in each of the coordinate planes. (The trace is the intersection of the given plane with a coordinate plane.) The trace, like every line, is described by two first degree equations, but one of these is the equation of a coordinate plane, i.e.. $x = 0$ or $y = 0$ or $z = 0$.

Some Generalizations About Planes and Their Equations--Suggestions for Constructing Problems.

We are now in a position to make certain general observations:

1. Any plane has infinitely many equations.

If a given plane is represented by the equation

$$Ax + By + Cz + D = 0 ,$$

it is also represented by

$$k(Ax + By + Cz + D) \quad 0 ,$$

where k is any non-zero constant.

The proof of this may be either algebraic or geometric:

(a) every number triple satisfying either equation satisfies the other; or

(b) the traces of the two planes are identical.

The converse is also true. Equations in which the coefficients are proportional represent coincident planes. For if two planes have equations

$$A_1x + B_1y + C_1z + D_1 = 0$$

$$A_2x + B_2y + C_2z + D_2 = 0$$

and

$$\frac{A_1}{A_2} = \frac{B_1}{B_2} = \frac{C_1}{C_2} = \frac{D_1}{D_2} = k$$

then the first equation is k times the second, and the equations represent the same plane.

These results are useful throughout our algebraic study. For example, if we have a system of two equations, in which one equation is a multiple of the other, we know that the second equation contributes no information not already given by the first. Thus, a point whose coordinates satisfy the two equations still has the two degrees of freedom that characterize the point whose coordinates satisfy a single equation. This is the third case described under Degrees of Freedom for two equations.

2. If two planes have equations that can be reduced to the form

$$Ax + By + Cz = D$$

where $D \neq F$

$$Ax + By + Cz = F$$

the planes are parallel, and there is no common solution.

Again, the proof may be algebraic or geometric:

(a) If (x_0, y_0, z_0) is any number triple, it cannot satisfy both these equations since

$$Ax_0 + By_0 + Cz_0$$

cannot be equal to both D and F if $D \neq F$; or

(b) the traces of the planes are parallel lines.

3. These two cases are summarized in the following rule:

<u>If corresponding coefficients of two first degree equations are proportional, then their graphs</u>

(a) <u>are the same plane if the constant terms have the same ratio as the coefficients,</u>

(b) <u>are parallel planes if their constant terms are not in the same ratio as the coefficients.</u>

This information gives us a way to recognize at a glance two equations that are inconsistent or dependent. It also gives the teacher the ability to make up problems with great ease. He need merely put down any left member of first degree and any constant term for the first equation. For the first case, double the first equation, triple it, or transpose some terms. For example:

Start with the equation $3x + 5y - z = 7$.

Double the equation and transpose the y term

$$6x - 2z = 14 - 10y .$$

The resulting equation represents the same plane.

For the second case, copy down the same left member for the equation but make its constant term different:

$$3x + 5y - z = 12.$$

This equation represents a plane parallel to the first.

A more sophisticated version of this procedure involves doubling, trebling, or multiplying the left member by -1 while taking care to do something else with the constant.

4. <u>Conversely, two planes meet in a line if and only if their corresponding coefficients are not proportional.</u>

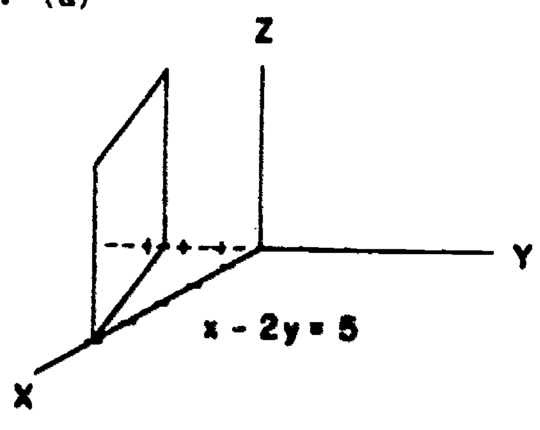
Again, examples of this sort can be invented in the time it takes to write them down: take any first degree equation as the first, and change the coefficients for the second one somehow, so that they are not proportional to the first ones. Having accomplished this much one is safe. Any constant term whatever will do. New coefficients not proportional to the first ones can be obtained in many ways. For example: keep one of them the same and change some other one; or add one to each of them; or change some of the signs, but not all, etc. We give a collection of such equations:

$$3x + 5y - z = 7$$
$$3x + 5y + z = 7$$
$$4x - 5y + 2z = 5 \text{ , etc.}$$

The four results stated above may be made the basis of a preliminary examination of a system of equations. If we can tell by inspection that two of the given equations are inconsistent, we know immediately that the system has no solution. If we can tell by inspection that one of the given equations is dependent on the others, we know that the number of degrees of freedom is larger than would be in the case if all the equations were independent.

Exercises 4. - Answers.

1. (a)



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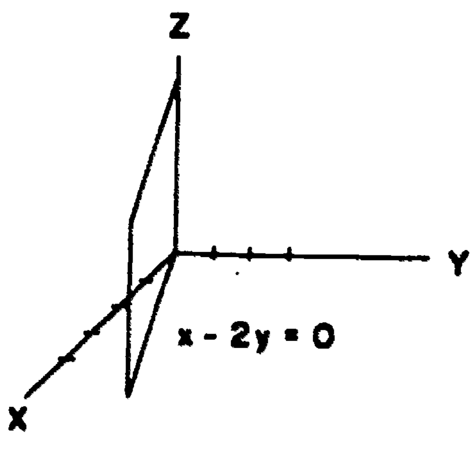

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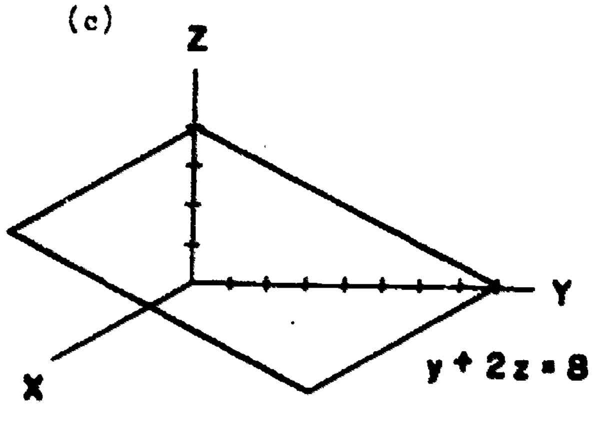

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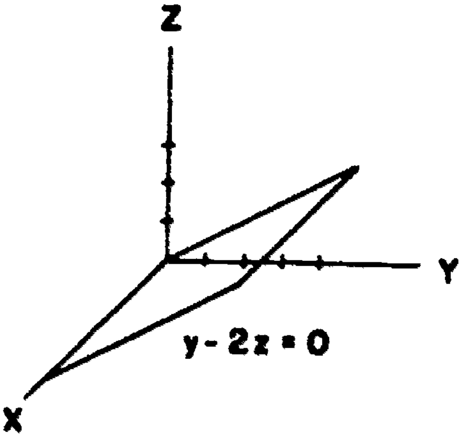
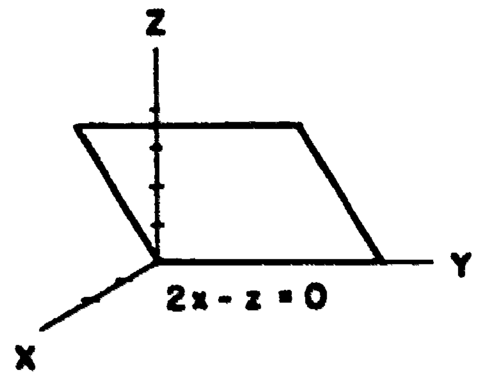
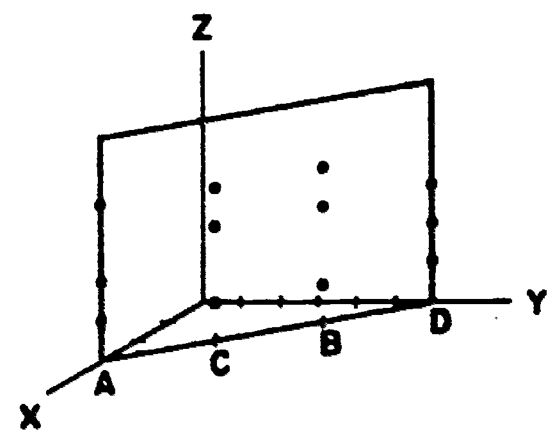


(e)



2. $2x + y = 6$



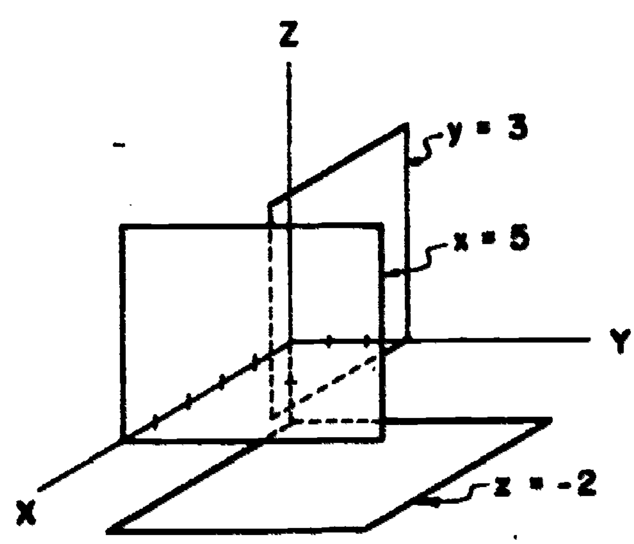
A(3,0,0). Also on the graph are (3,0,1), (3,0,2), (3,0,4).

B(1,4,0). Also on the graph are (1,4,1), (1,4,3), (1,4,4).

C(2,2,0). Also on the graph are (2,2,1), (2,2,3), (2,2,4).

D(0,6,0). Also on the graph are (0,6,1), (0,6,2), (0,6,3).

3.



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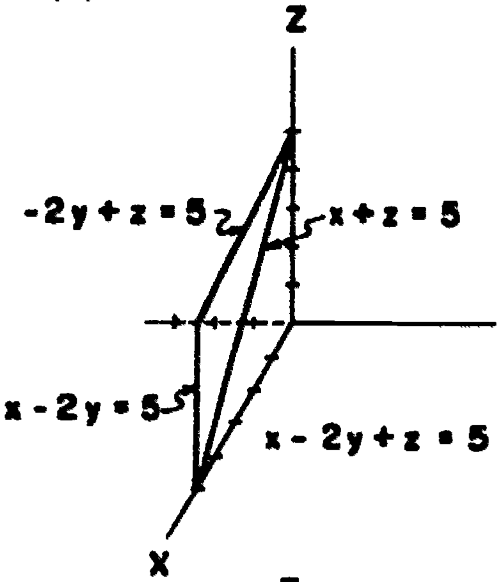
Exercises 5. - Answers.

1. (a)



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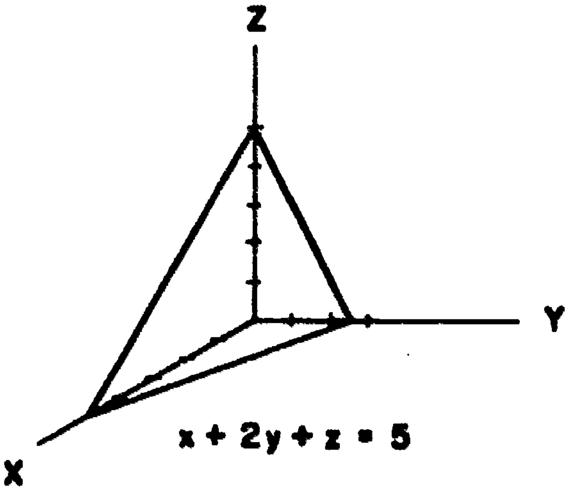

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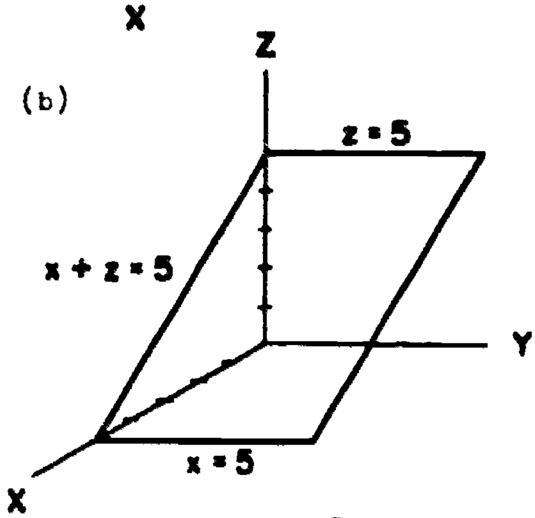
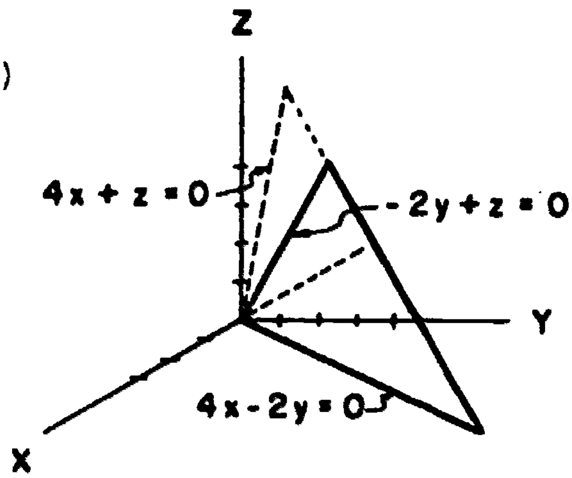


x + z = 5

(e)



4x - 2y + z = 0

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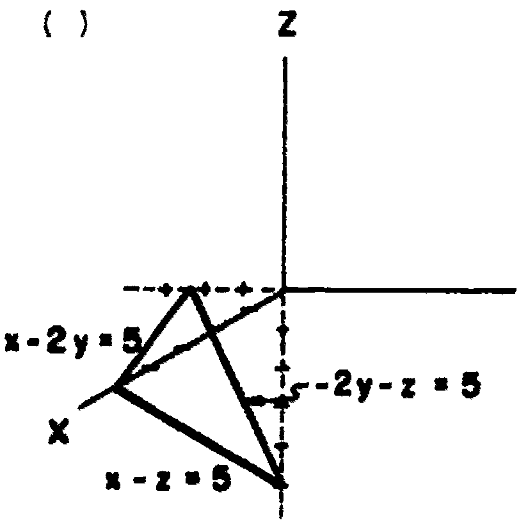
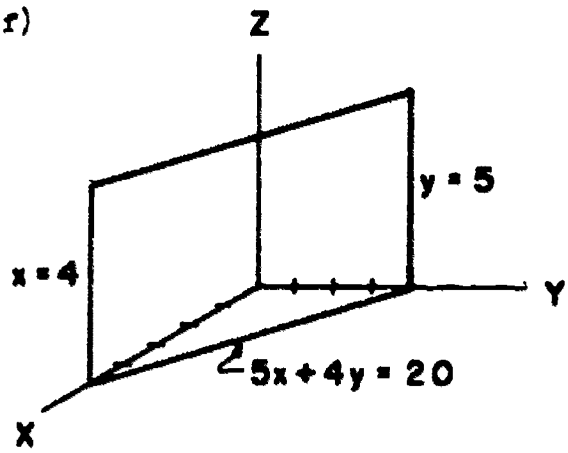


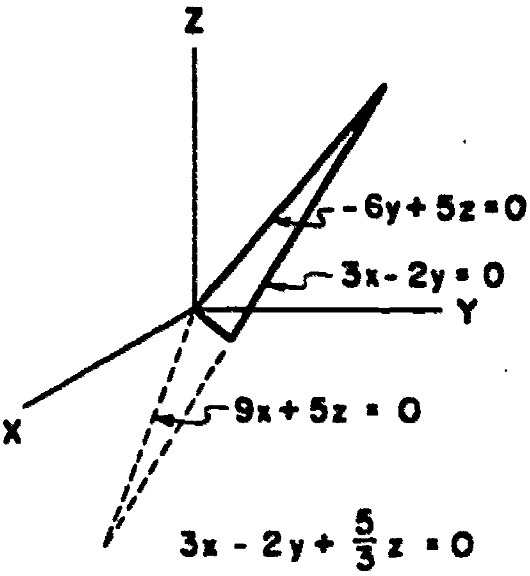
x - 2y - z = 5

(f)

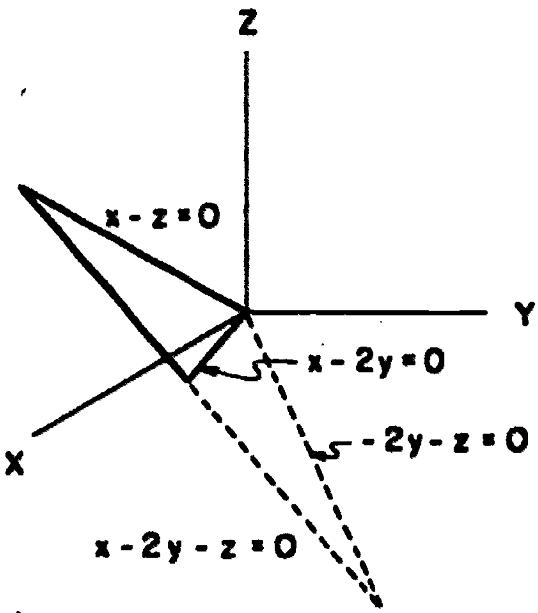


5x + 4y = 20

(g)



(i)



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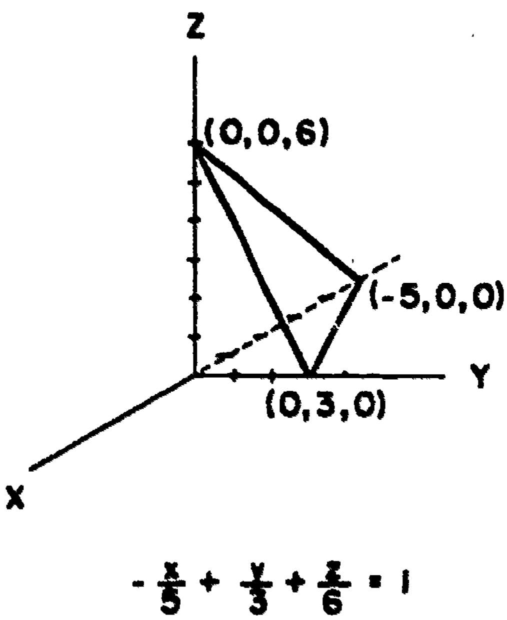
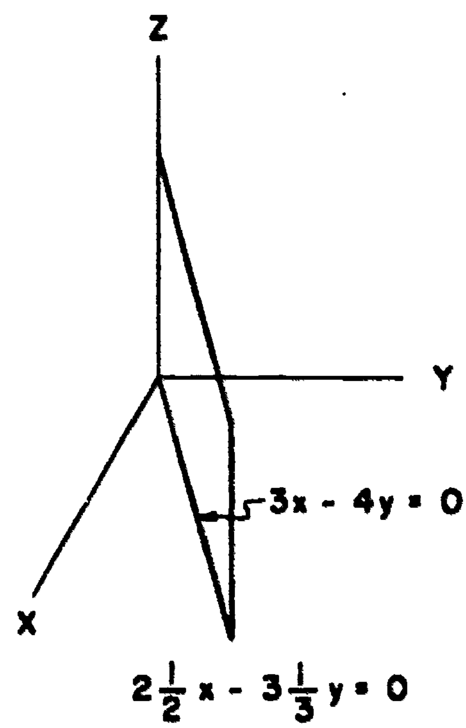


(j)



7. <u>Comment</u>.

The geometric information presented in Section 4 and 5 is sufficient for the minimal purposes of the pamphlet. Section 7 presents additional geometric information that will throw light on the later work and will be of interest to those students who enjoy three-dimensional studies. If the teacher finds time to include this material, and if some students find the graphing of space figures excessively difficult, it may be helpful if such assignments are made to groups of 2 or 3 students. If Section 7 is omitted, <u>the teacher should be sure to teach the material presented in Examples 2 and 3 of</u>

the section. This is covered by the discussion summarized in point (3) above. (Page 3.)

Parametric Representation of the Line of Intersection of Two Intersecting Planes.

(a) The line intersects all the coordinate planes. Once we have disposed of systems of 2 equations in 3 variables in which the planes are coincident or parallel, we must undertake the more formidable problem of representing the line of intersection of planes that do intersect. Actually, this line is represented by the two equations of the given intersecting planes; but since we know from our discussion above that the point describing the line has a single degree of freedom, we seek a representation of the line in which the three coordinates of the point are described in terms of a single parameter. We seek to describe the coordinates of any point on the line as functions of a single variable--this is the variable to which we can assign arbitrary values in finding as many points as we want in the solution set. Our manipulation of the given equations is aimed at expressing all three variables in terms of one of them so that the variable that is arbitrary is clearly indicated. In the non-special case, when the line of intersection cuts all three coordinate planes, any one of the three variables may be chosen arbitrarily, so that there are three different parametric representations of the line, one in which x is arbitrary, one in which y is arbitrary, and one in which z is arbitrary. To derive each of these, we find the equations of a pair of planes through the line, each equation having in it one variable with a zero coefficient. This is achieved by eliminating each variable in turn from the given equations, and combining the resulting three equations, two by two. For example, an equation containing only x and y (the coefficient of z is zero) is combined with an equation containing only x and z. Since x and y are in the first equation, y can be expressed in terms of x. Since x and z are in the second equation, z can be expressed in terms of x. In this case, x serves a parameter. (Note that geometrically the planes corresponding to equations that have a single zero coefficient are parallel to an axis, e.g., Equation (8d) is $x + 3z + 1 = 0$; this plane is parallel to the y-axis.)

(b) The line is perpendicular to one of the coordinate planes. If the line we seek to describe is perpendicular to one of the coordinate planes the situation is special. For example, if the line is perpendicular to the XY-plane, the plane that passes through it and is parallel to the x-axis is also parallel to the XZ-plane. Similarly, the plane that passes through the given line, parallel to the y-axis, is also parallel to the YZ-plane. Indeed any plane that passes through the given line is parallel to the z-axis. Thus, the coordinates of a point on the line have a very special parametric representation, namely,

$x = a$ (y and z have zero coefficients)

$y = b$ (x and z have zero coefficients)

z is arbitrary.

(c) The line is parallel to one of the coordinate planes. In a similar way, we find that if the line we seek to describe is parallel to one coordinate plane, but intersects the other two, the situation is special. Here, one variable will be constant, but either of the others may be expressed in terms of the third. This case is discussed in Example 2 of Section 8.

The Method of Elimination.

The justification for the familiar procedure used in eliminating one variable from the given equations is the theorem that is illustrated for a special case in starred Section 10. This argument is the same as that for equations in two variables. It establishes the fact that the solution set for any given system

$$\begin{cases} f_1 = 0 , \\ f_2 = 0 ; \end{cases}$$

is the same as for a new system

$$\begin{cases} f_1 = 0 , \\ a_1 f_1 + a_2 f_2 = 0 . \quad (a_1 \text{ and } a_2 \text{ not both zero}) \end{cases}$$

or for a second system

$$\begin{cases} a_1 f_1 + a_2 f_2 = 0 , & \text{Eq. (1)} \\ b_1 f_1 + b_2 f_2 = 0 . \quad (b_1 \text{ and } b_2 \text{ not both zero}) & \text{Eq. (2)} \end{cases}$$

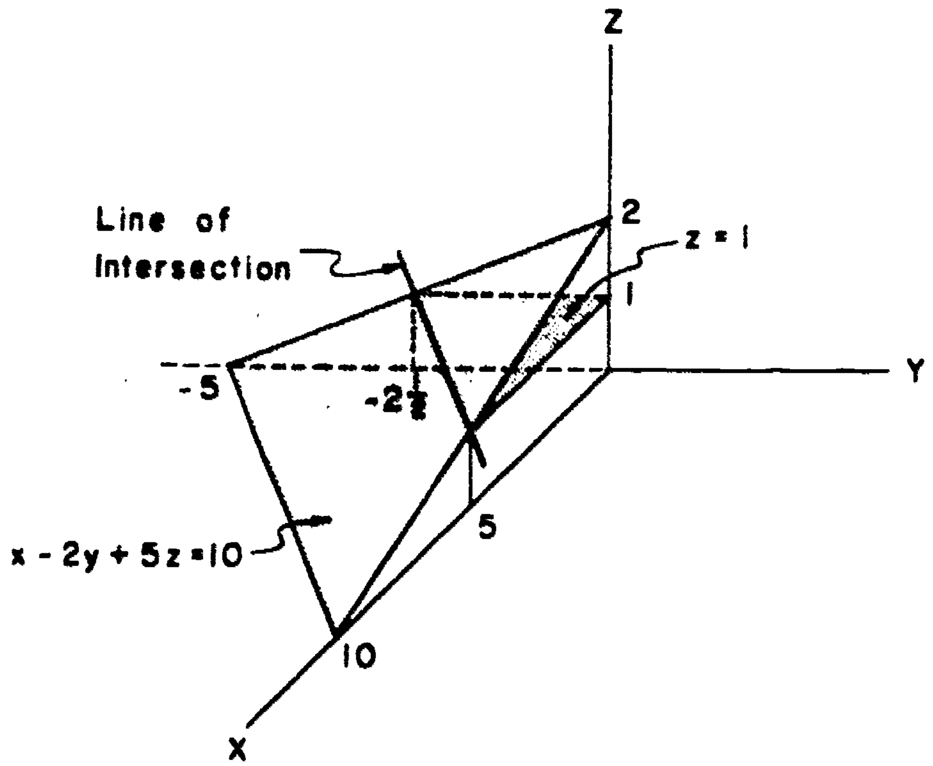
The two new systems are thus equivalent to the given system. The expressions in the left members of Equations (1) and (2) are linear combinations of f_1 and f_2. Equations (1) and (2) represent planes through the line of intersection of the planes of the given system if these intersect; they represent planes parallel to the given planes if these are parallel; they represent the same plane if the original equations did. It is the case for intersecting planes that is studied for a special pair of equations in Section 10. (The teacher is urged to study this section, even if it is not covered in class, to gain some familiarity with these ideas.)

Exercises 7. - Answers.

1. $x - 2y + 5z = 10$

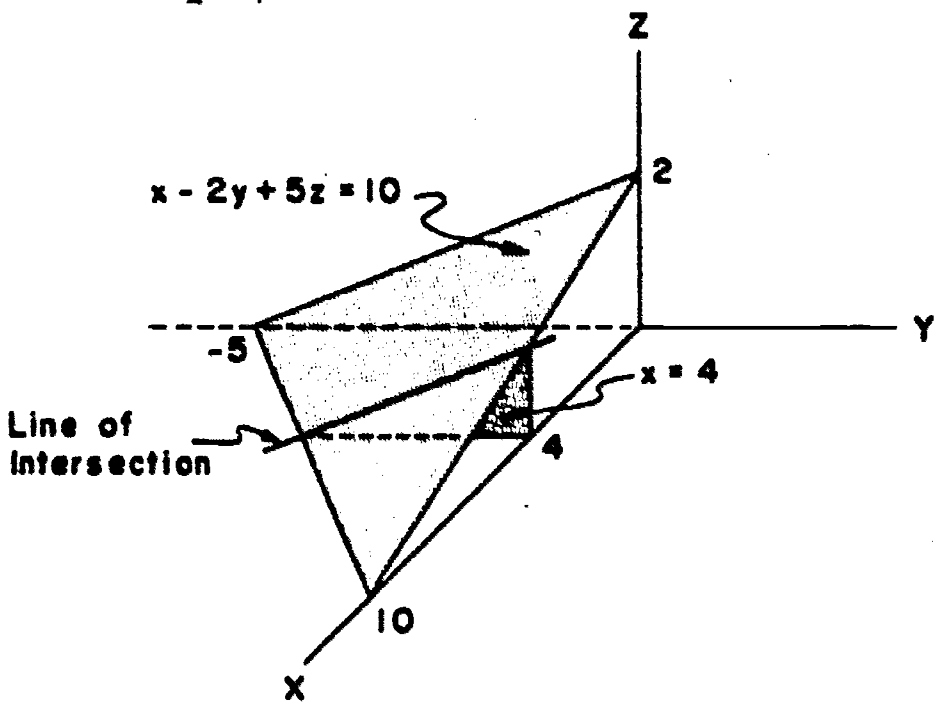
   Intercepts: $(10,0,0)$
   $(0,-5,0)$
   $(0,0,2)$

   $Z = 1$ Parallel to XY-plane.



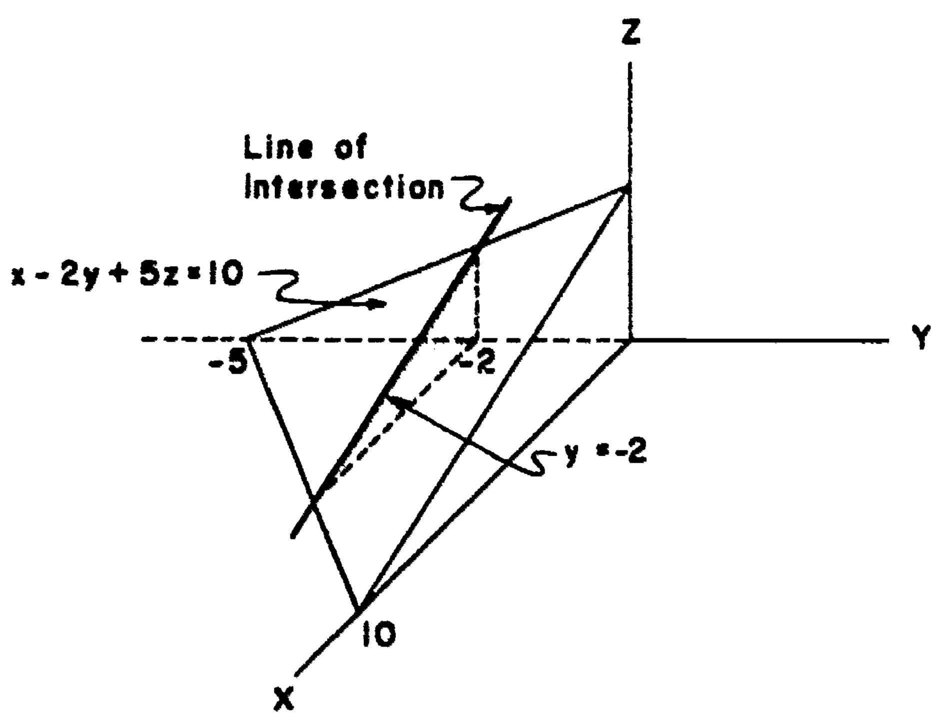
2. $x - 2y + 5z = 10$

$x = 4$



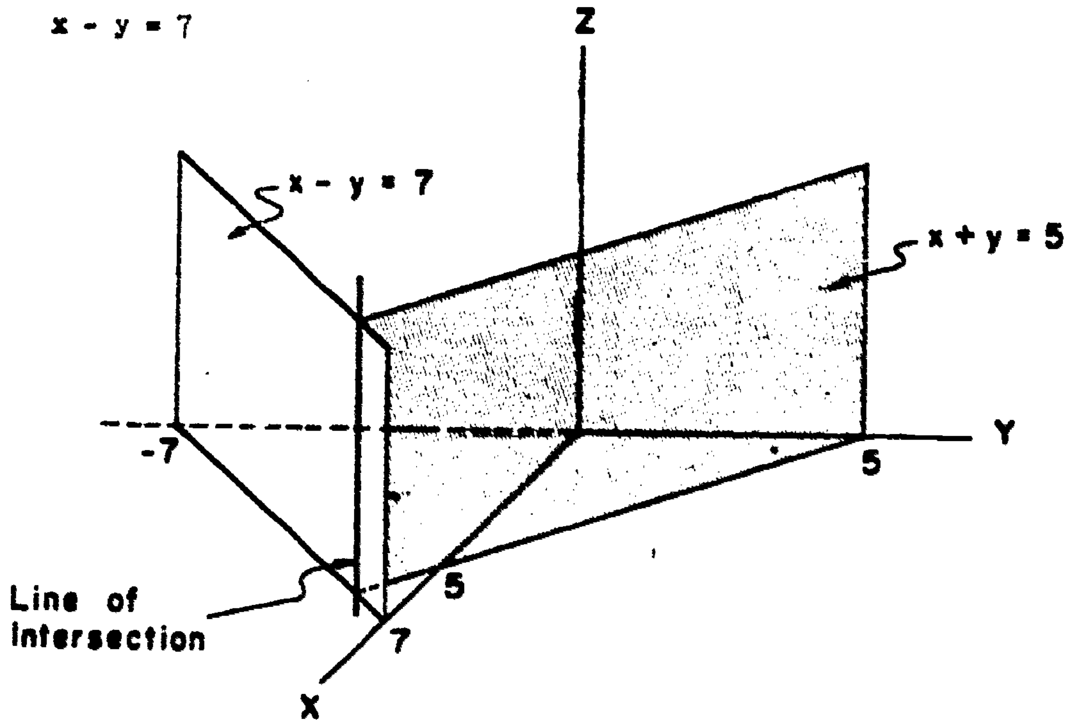
3. $x - 2y + 5z = 10$

$y = -2$



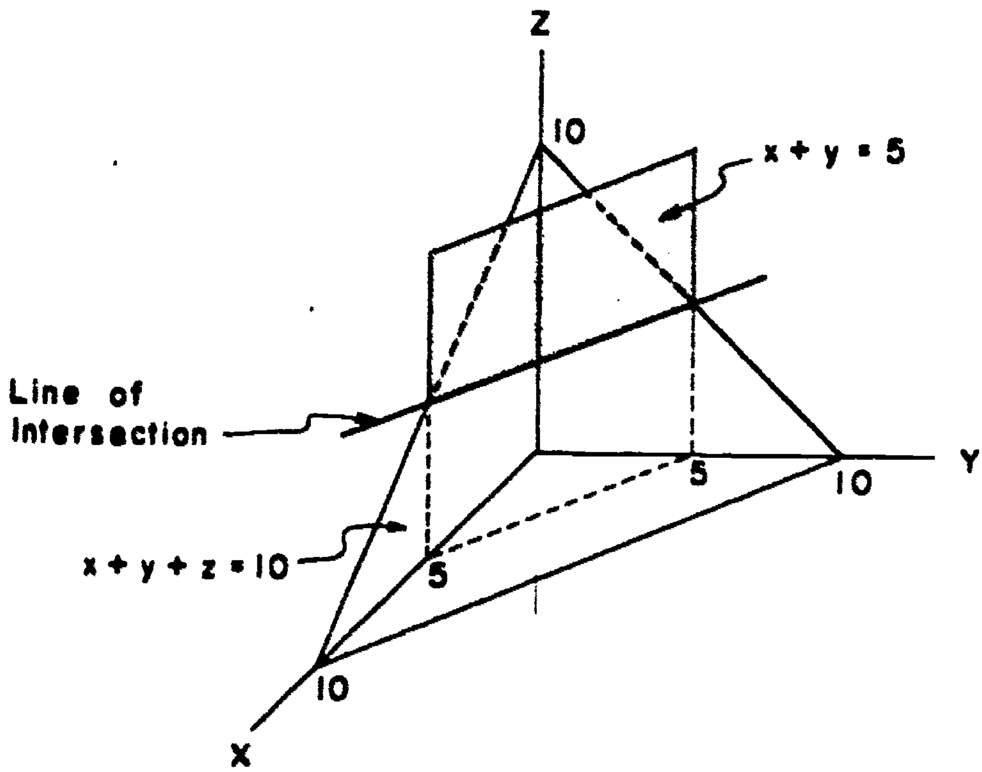
4. $x + y = 5$

$x - y = 7$

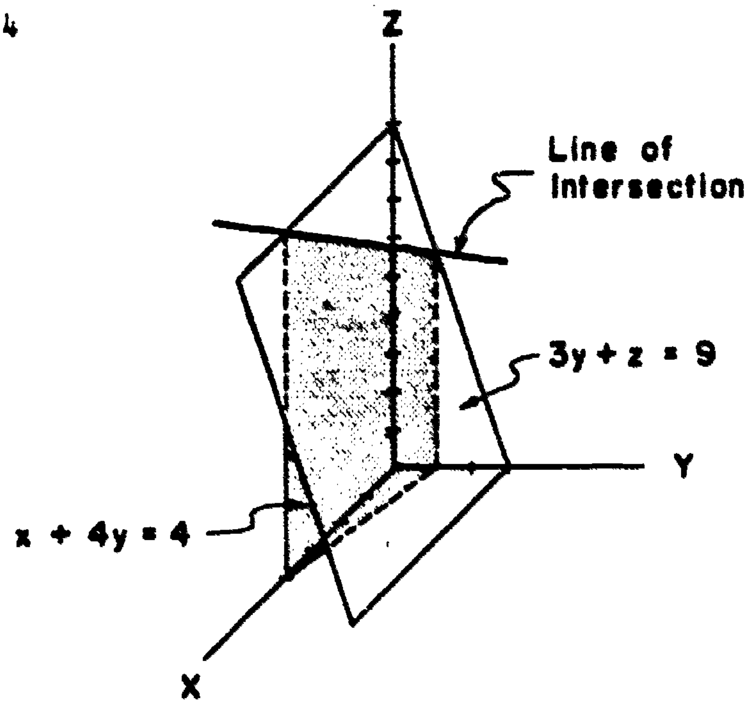


5. $x + y = 5$

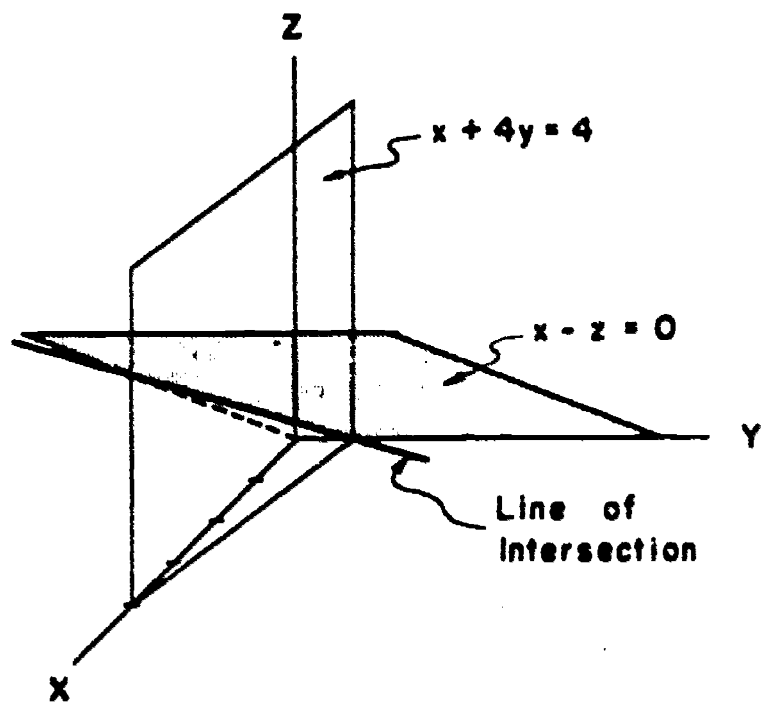
$x + y + z = 10$



6. $3y + z = 9$
   $x + 4y = 4$



7. $x + 4y = 4$
   $x - z = 0$

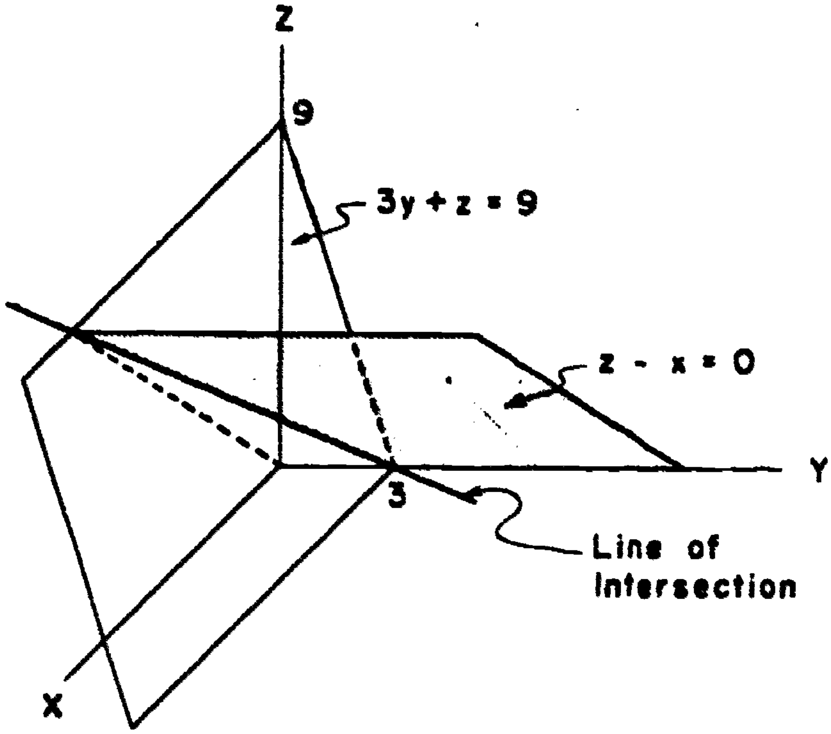


8. $3x + y - z = 2$
   $2z = 6x + 2y - 4$

Same plane

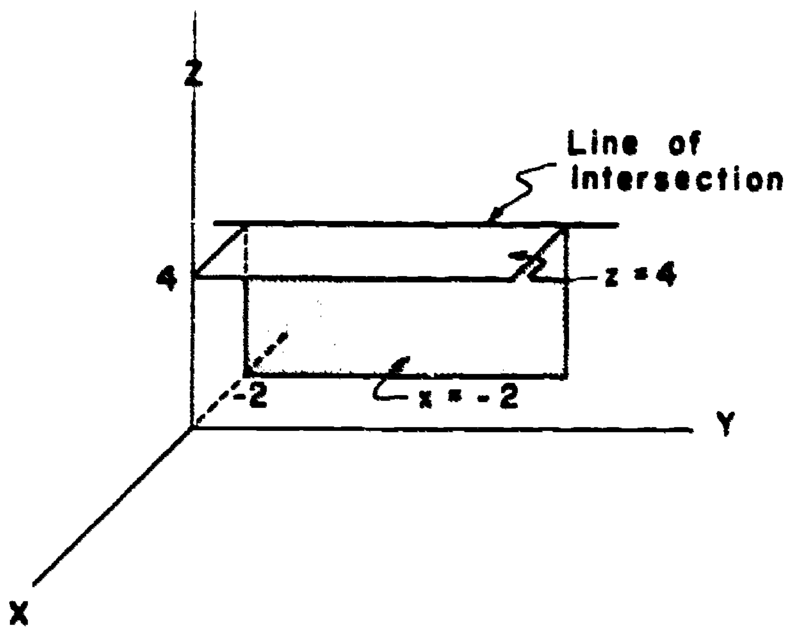
9. $z - x = 0$

$3y + z = 9$

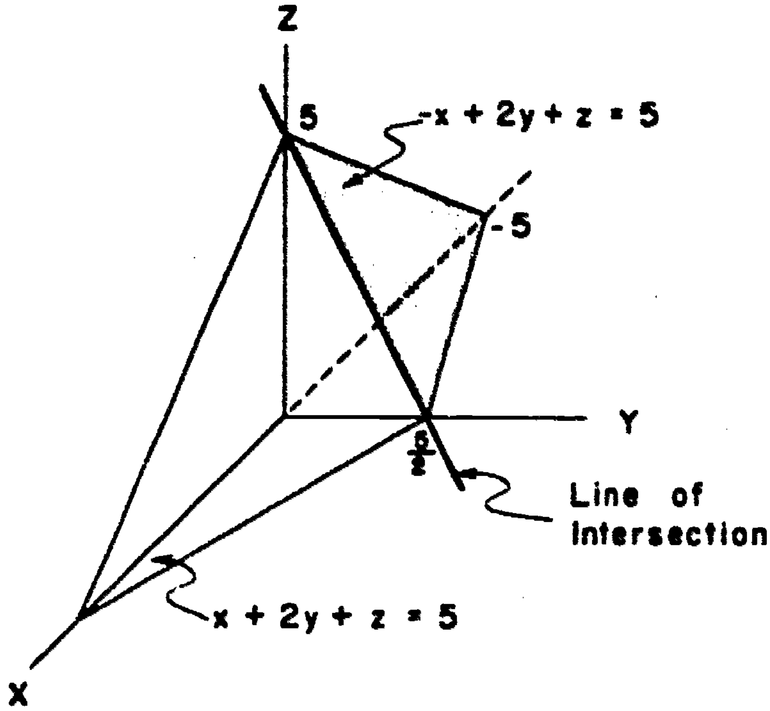


10. $x = -2$

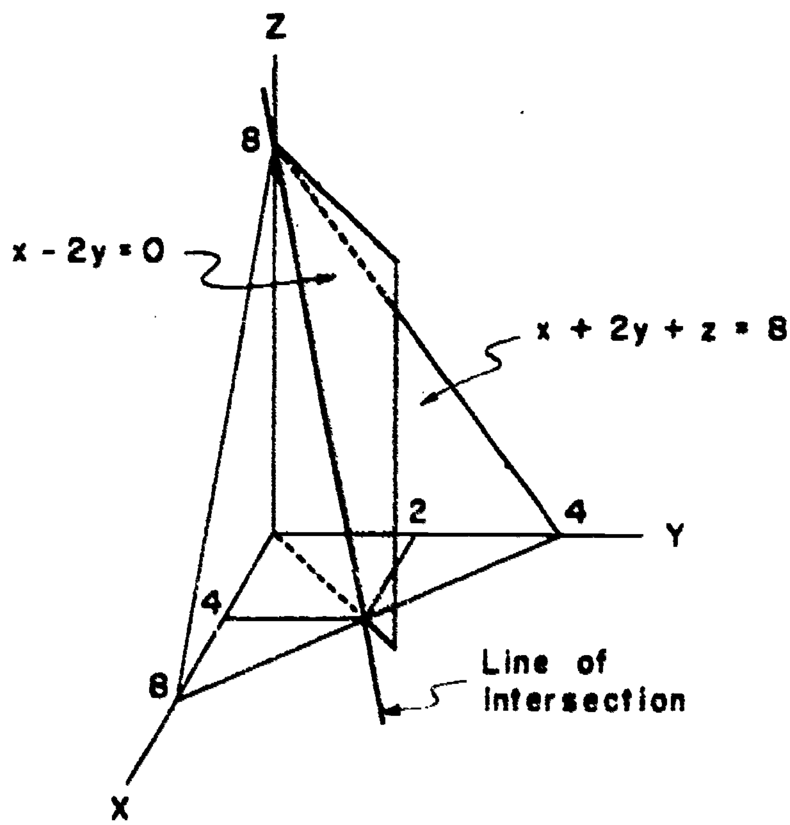
$z = 4$



11. $x + 2y + z = 5$
    $-x + 2y + z = 5$



12. $x + 2y + z = 8$
    $x - 2y = 0$



8. <u>Comments</u>.

In Section 8 the equations we obtain represent planes through the line of intersection we seek to describe. Therefore, when we use the new equations to write in parametric form the coordinates of a point on the line of intersection, we have actually described a point on the line of intersection of the <u>given</u> planes as well as on the line of intersection of the planes represented by the new equations.

<u>Exercises 8. - Answers</u>.

1. $x - 3y - z = 11$

$x - 5y + z = 1$

$2x - 8y = 12$

$-2y + 2z = -10$

$\therefore x = 4y + 6$

y arbitrary

$z = y - 5$

x	-2	6	14	22
y	-2	0	2	4
z	-7	-5	-3	-1

Check by substituting in given equations.

$4y + 6 - 3y - y + 5 \stackrel{?}{=} 11$

$4y + 6 - 5y + y - 5 \stackrel{?}{=} 1$

2. $x + 2y - z = 8$,

$x + y + z = 0$.

$y - 2z = 8$

$2x + 3y = 8$

$\therefore x = \frac{1}{2}(-3y + 8)$

y arbitrary

$z = \frac{1}{2}(y - 8)$

x	4	1	-2	-5
y	0	2	4	6
z	-4	-3	-2	-1

Check by substituting in the given equations.

$-3y + 8 + 4y - y + 8 \stackrel{?}{=} 16$

$-3y + 8 + 2y + y - 8 \stackrel{?}{=} 0$

3. $x + y - z = 5$

$x + 2y + z = 0$

$y + 2z = -5$

$2x + 3y = 5$

$x = \frac{1}{2}(-3y + 5)$

y arbitrary

$z = \frac{1}{2}(-y - 5)$

x	$\frac{5}{2}$	1	-2	-5
y	0	1	3	5
z	$-\frac{5}{2}$	-3	-4	-5

Check by substituting in the given equations.

$-3y + 5 + 2y + y + 5 \stackrel{?}{=} 10$

$-3y + 5 + 4y - y - 5 \stackrel{?}{=} 0$

4. $2x + 4y - 5z = 7$

$4x + 8y - 5z = 14$

$2x + 4y = 7$

$z = 0$

y arbitrary

$\therefore x = -2y + \frac{7}{2}$

x	$\frac{7}{2}$	$\frac{3}{2}$	$-\frac{1}{2}$	$-\frac{5}{2}$
y	0	1	2	3
z	0	0	0	0

Check by substituting in the given equations.

$-4y + 7 + 4y - 5(0) \stackrel{?}{=} 7$

$-8y + 14 + 8y - 5(0) \stackrel{?}{=} 14$

5. $-2x + y + 3z = 0$,

$-4x + 2y + 6z = 0$.

Since the second equation is twice the first, the two planes coincide. Thus, there is no line of intersection.

6. $2x + 6z - 18y = 6$

$x - 3z - y = -3$

$2x - 10y = 0$

$6z - 8y = 6$

$\therefore x = 5y$

y arbitrary

$z = \frac{1}{3}(4y + 3)$

x	-15	0	15	30
y	-3	0	3	6
z	-3	1	5	9

Check by substituting in the given equations.

$10y + 8y + 6 - 18y \stackrel{?}{=} 6$

$5y - 4y - 3 - y \stackrel{?}{=} -3$

7. $3x - 4y + 2z = 6$

$6x - 8y + 4z = 14$.

If we divide both members of the second equation by 2 , we obtain

$$3x - 4y + 2z = 7$$

We can see by inspection that no number triple can be in the solution set of both these equations. Therefore, the corresponding planes have no point in common. The planes are parallel.

8. $-5x + 4y + 8z = 0$

$-3x + 5y + 15z = 0$

$13x + 20z = 0$

$13y + 51z = 0$

$x = -\frac{20}{13}z$

$y = -\frac{51}{13}z$

z arbitrary

x	20	0	-20	-40
y	51	0	-51	-102
z	-13	0	13	26

Check by substituting in the given equations.

$\frac{100}{13}z - \frac{204}{13}z + 8z \stackrel{?}{=} 0$

$\frac{60}{13}z - \frac{255}{13}z + 15z \stackrel{?}{=} 0$

9. $4x - 7y + 6z = 13$,

$5x + 6y - z = 7$.

$-59y + 34z = 37$

$34x + 29y = 55$

$x = \frac{1}{34}(-29y + 55)$

y arbitrary

$z = \frac{1}{34}(59y + 37)$

x	$\frac{55}{34}$	$\frac{13}{17}$	$\frac{42}{17}$	$-\frac{3}{34}$
y	0	1	-1	2
z	$\frac{37}{34}$	$\frac{48}{17}$	$\frac{-11}{17}$	$\frac{155}{34}$

10. $-10x + 4y - 5z = 20$

$2x - \frac{4}{5}y + z = 4$

If we multiply both members of equation 2 by -5 , we obtain

$$-10x + 4y - z = -20$$

We can see by inspection that no number triple can be in the solution set of both these equations. Therefore, the corresponding planes have no point in common. The planes are parallel.

---

9. <u>Comments</u>. <u>Application of the Method of Elimination</u>.

The same idea dominates Section 9. Consider an example studied there. We discussed the system (Example 1).

$$x + 2y - 3z = 9$$
$$2x - y + 2z = -8$$
$$-x + 3y - 4z = 15$$

and converted into the equivalent system

$$\begin{aligned} x + 2y - 3z &= 9 \\ -5y + 8z &= -26 \\ z &= -2 \end{aligned}$$

by repeated application of exactly the same technique used in Section 8: selecting two of our equations and playing them off against one another to get rid of variables one at a time. Even the final stage of the discussion of Example 1 is an instance of the same process. We arrive eventually at the system

$$\begin{aligned} x &= -1 \\ y &= 2 \\ z &= -2 \end{aligned}$$

by subtracting appropriate multiples of the last equation from the first two, eliminating z, and then using the second equation to get y out of the first. This leaves us with the last system given above which is equivalent to the original system. The last system is so extraordinarily simple that we can read off its solution set at a glance.

A Systematic Method for Studying Three Equations in Three Variables.

The problem discussed in Section 9 is the most complicated case we consider with three variables--the case in which there are as many equations as variables. Figure 9b illustrates the eight essentially different configurations formed by three planes in space. These pictures are included only for their interest. It is not important at this point that the student understand all the details.

With three planes there are four different types of solution sets (there were only three in the case of two planes):

1. The empty set
2. A single point
3. A line
4. A plane

The main business of Section 9 is the presentation of a systematic algebraic method for determining everything there is to know about systems of first degree equations: whether there are any solutions and how to find all of them. This method, "elimination", is applicable to systems having any number of equations and any number of variables. It is spelled out in detail only for three equations in three variables, since this case is probably the smallest one complicated enough to be of any real interest. Restricting ourselves to this case, we give examples to illustrate the application of the method not only to the type of system in which the solution set consists of a single number triple, but also to several types in which the systems are inconsistent or dependent.

This method (sometimes called triangulation) is attributed to Gauss (1777-1855), the greatest mathematician since Newton. It gives the student the basic point of view he will need if he goes on to work in a large computing center. Its popularity reflects the fact that it gives an orderly procedure for handling systems of linear equations which, for many important cases, involves substantially fewer arithmetic operations than other methods. Using this method, we have the system essentially solved by the time we discover whether or not the solution is unique.

Relation of Method of Elimination to Cramer's Rule.

Those familiar with Cramer's rule (this is discussed in many of the older texts on College Algebra; it is usually not included in the newer texts) may be interested in its relation to the method of Gauss that we have presented. Observe first that, if the given equations are

$$\begin{aligned} A_1x + B_1y + C_1z &= D_1 \\ A_2x + B_2y + C_2z &= D_2 \\ A_3x + B_3y + C_3z &= D_3 \end{aligned}$$

then our "triangulation" method replaces the given system by

$$\begin{cases} A_1x + B_1y + C_1z = D_1 \\ \begin{vmatrix} A_1 B_1 \\ A_2 B_2 \end{vmatrix} y + \begin{vmatrix} A_1 C_1 \\ A_2 C_2 \end{vmatrix} z = \begin{vmatrix} A_1 D_1 \\ A_2 D_2 \end{vmatrix} \\ \begin{vmatrix} A_1 B_1 \\ A_3 B_3 \end{vmatrix} y + \begin{vmatrix} A_1 C_1 \\ A_3 C_3 \end{vmatrix} z = \begin{vmatrix} A_1 D_1 \\ A_3 D_3 \end{vmatrix} \end{cases}$$

and then by

$$\begin{aligned} &A_1x + B_1y + C_1z = D \\ &\begin{vmatrix} A_1 B_1 \\ A_2 B_2 \end{vmatrix} y + \begin{vmatrix} A_1 C_1 \\ A_2 C_2 \end{vmatrix} z = \begin{vmatrix} A_1 D_1 \\ A_2 D_2 \end{vmatrix} \end{aligned}$$

$$\begin{vmatrix} \begin{vmatrix} A_1B_1 \\ A_2B_2 \end{vmatrix} & \begin{vmatrix} A_1C_1 \\ A_2C_2 \end{vmatrix} \\ \\ \begin{vmatrix} A_1B_1 \\ A_3B_3 \end{vmatrix} & \begin{vmatrix} A_1C_1 \\ A_3C_3 \end{vmatrix} \end{vmatrix} z = \begin{vmatrix} \begin{vmatrix} A_1B_1 \\ A_2B_2 \end{vmatrix} & \begin{vmatrix} A_1D_1 \\ A_2D_2 \end{vmatrix} \\ \\ \begin{vmatrix} A_1B_1 \\ A_3B_3 \end{vmatrix} & \begin{vmatrix} A_1D_1 \\ A_3D_3 \end{vmatrix} \end{vmatrix}$$

The last equation can be shown to be equivalent to

$$A_1 \begin{vmatrix} A_1B_1C_1 \\ A_2B_2C_2 \\ A_3B_3C_3 \end{vmatrix} z = A_1 \begin{vmatrix} A_1B_1D_1 \\ A_2B_2D_2 \\ A_3B_3D_3 \end{vmatrix}$$

This is one of the equations derived by applying Cramer's rule. But for practical computing, the "elimination" or "triangulation" method has the great advantage that the nature of the solution becomes clear at this point; if it is unique, we find the solution with a minimum of additional computation. The mastery of this method should be a principal objective in teaching the material of this pamphlet.

Exercises 9. - Answers.

1. (3,4,5)
2. (2,3,3)
3. (2,-1,1)
4. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equation

$$x = 3z + 5$$
$$y = 2z + 4$$
$$z \ \text{arbitrary}$$

5. (1,-1,1)
6. (1,-1,2)
7. The three planes coincide. The solution set is an infinite set of triples corresponding to all the points in the plane.
8. (-1,-2,3)
9. The system is inconsistent. The solution set is empty.

10. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = 2z$$
$$y = \frac{1}{2}z$$
$$z \text{ arbitrary}$$

11. $(4,6,3)$
12. $(1,2,-1)$
13. The system is inconsistent. The solution set is empty.
14. The system is inconsistent. The solution set is empty.
15. $(\frac{1}{3}, \frac{1}{2}, 1)$
16. The three planes have a line in common. (The second equation represents the same plane as the first equation.) The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{9}{2} - y$$
$$y \text{ arbitrary}$$
$$z = -\frac{3}{2}$$

17. $(\frac{3}{4}, \frac{1}{2}, 3)$
18. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{1}{5}(-7z + 17)$$
$$y = \frac{1}{5}(z - 1)$$
$$z \text{ arbitrary}$$

19. $(\frac{1}{3}, -\frac{2}{5}, \frac{1}{2})$
20. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = -\frac{1}{7}(6z - 5)$$
$$y = +\frac{1}{7}(16z - 11)$$
$$z \text{ arbitrary}$$

21. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = \frac{1}{7}(-z + 2)$$
$$y = \frac{1}{7}(-5z + 17)$$
$$z \text{ arbitrary}$$

22. The three planes have a line in common. The solution set is an infinite set of triples corresponding to the points on this line, and described by the equations,

$$x = -7z - 10$$
$$y = -5z - 6$$
$$z \text{ arbitrary}$$

23.

Food	Vitamin Content		
	A	B	C
I	1	3	4
II	2	3	5
III	3	0	3
Requirements	11	9	20

If we buy x units of I, y of II, 20 of III, we want

$$\begin{aligned} x + 2y + 3z &= 11\ , \\ 3x + 3y &= 9\ , \\ 4x + 5y + 3z &= 20\ ; \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + 3z &= 11\ , \\ x + y &= 3\ , \\ 4x + 5y + 3z &= 20\ . \end{aligned}$$

Eliminate x :

$$\begin{aligned} x + 2y + 3z &= 11\ , \\ -y - 3z &= -8\ , \\ -3y - 9z &= -24\ ; \end{aligned} \qquad \text{or} \qquad \begin{aligned} x + 2y + 3z &= 11\ , \\ y + 3z &= 8\ , \\ y + 3z &= 8\ . \end{aligned}$$

Answer for (a): No--Our conditions are dependent.

for (b): Consider the system,

$$\begin{aligned} x + 2y + 3z &= 11\ , \\ y + 3z &= 8\ , \\ 6x + y + z &= 10\ . \end{aligned}$$

Eliminate x:

$$\begin{aligned} x + 2y + 3z &= 11 , \\ y + 3z &= 8 , \\ 11y + 17z &= 56 . \end{aligned}$$

Eliminate y:

$$\begin{aligned} x + 2y + 3z &= 11 , \\ y + 3z &= 8 , \\ 16z &= 32 . \end{aligned}$$

Thus, $z = 2$, $y = 8 - 3z = 2$, $x = 11 - 2y - 3z = 1$.

Answer for (b): Yes. 1 unit of I and 2 each of II, III.

24.

$$\begin{aligned} x + y - 5 &= 0 , \\ -x + 3z - 2 &= 0 , \\ x + 2y + z - 1 &= 0 , \\ y + z + 4 &= 0 . \end{aligned}$$

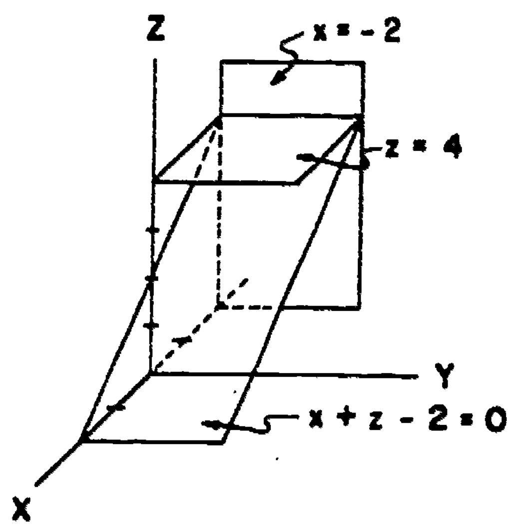
Suppose we apply the standard procedure given in Section 9, if only to see what happens. We eliminate x by subtracting appropriate multiples of the first equation from others:

$$\begin{aligned} x + y - 5 &= 0 , \\ y + 3z - 7 &= 0 , \\ y + z + 4 &= 0 , \\ y + z + 4 &= 0 . \end{aligned}$$

We have found that, in our original system, the first, third and fourth equations are dependent; indeed

$$x + y - 5 = 1(x + 2y + z - 1) - 1(y + z + 4) .$$

These three therefore all meet in a line. Since we were <u>given</u> the fact that the system has only one solution triple, this line must pierce the second plane in a single point. Hence any one of the four equations <u>except for the second</u> may be omitted, the line being determined by any pair of the three planes containing it.



<u>Exercises 10. - Answers.</u>

1. (a) $a(x + 2) + b(z - 4) = 0$

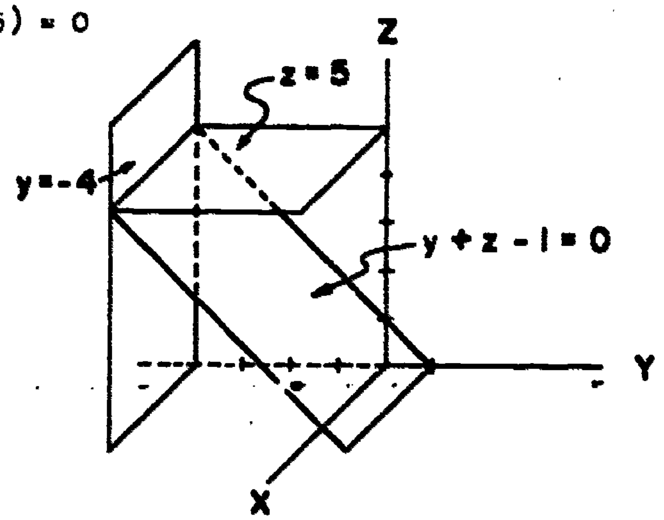
$a = 1$, $b = 1$;

$x + z - 2 = 0$

(b) $a(y + 4) + b(z - 5) = 0$

$a = 1$, $b = 1$;

$y + z - 1 = 0$



2. (a) $a(x + 2y - 3z) + b(x - y + z - 1) = 0$

Substituting (1,2,1) for (x,y,z):

$a(1 + 4 - 3) + b(1 - 2 + 1 - 1) = 0$

$2a - b = 0$; $b = 2a$

Take $a = 1$, $b = 2$;

$(x + 2y - 3z) + 2(x - y + z - 1) = 0$

or $3x - z = 2$

(b) $a(2y - 3z - 2) + b(x + y + z) = 0$

Substituting (3,-1,0) for (x,y,z):

$a(-2 + 0 - 2) + b(3 - 1 + 0) = 0$

$-4a + 2b = 0$

$2a = b$

Take $a = 1$, $b = 2$

$(2y - 3z - 2) + 2(x + y + z) = 0$

$2x + 4y - z - 2 = 0$

(c) $a(x + z) + b(2x - y + z - 8) = 0$

Substituting (0,0,0) for (x,y,z):

$-8b = 0$; $b = 0$.

The equation is $x + z = 0$.

This shows that the plane represented by the first equation is the only plane through the given line of intersection that also passes through the origin.

(d) $a(2x - y + z - 3) + b(x - 3y + 4) = 0$

Substituting (2,2,1) for (x,y,z)

$a(4 - 2 + 1 - 3) + b(2 - 6 + 4) = 0$

$0 = 0$

For all values of a and b the plane

$a(2x - y + z - 3) + b(x - 3y + 4) = 0$

passes through the point $(2,2,1)$. This is because the given point lies on the line of intersection of the given planes.

*3. Since the second equation can be written

$$3(2x - y + 3z) = 5$$

it is clear that any triple in the solution set of the first equation (and therefore reducing the parenthesis to 1) will not be in the solution set of the second equation. Similarly, for any triple in the solution set of the second equation. Thus, the planes have no point in common and are parallel.

The equation (10e)

$$a(2x - y + 3z - 1) + b(6x - 3y + 9z - 5) = 0$$

can be written

$$(a + 3b)(2x - y + 3z) + (-a - 5b) = 0$$

If this plane passes through a point on the first plane, we know that

$$2x - y + 3z = 1$$

Therefore

$$a + 3b - a - 5b = 0$$
$$-2b = 0$$
$$b = 0$$

Thus, any plane represented by (10e) that passes through a point in the first plane must coincide with the first plane. Similarly, if a plane representec by (10e) passes through a point of the second plane, we have

$$(a + 3b)(\tfrac{5}{3}) + (-a - 5b) = 0$$
$$2a = 0$$
$$a = 0$$

Therefore, the plane coincides with the second plane.

We conclude that if $a \neq 0$ and $b \neq 0$, any plane represented by (10e) has no point in common with either of the given planes. It is therefore parallel to these planes.

* 4. $a(x + y - 3) + b(z - 4) = 0$
Substituting $(1,-1,1)$ for (x,y,z)
$a(-3) + b(-3) = 0$; $a = -b$.

Take $a = 1$, $b = -1$

$x + y - z + 1 = 0$

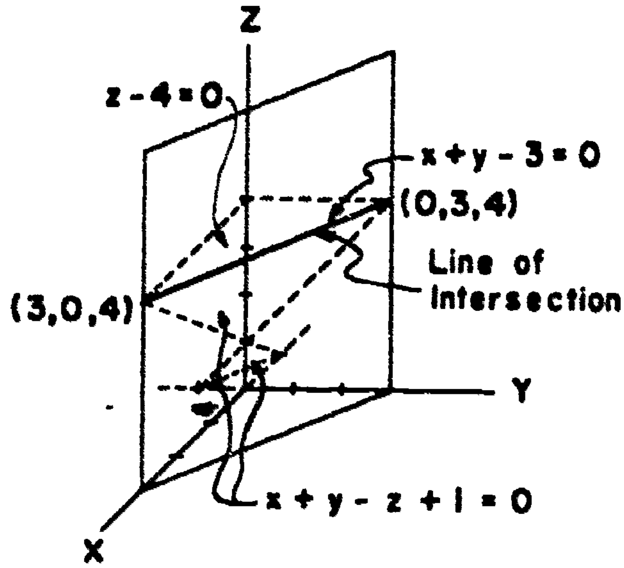
The trace of this plane in the XZ-plane is $x - z + 1 = 0$. This line intersects the XZ-trace of

$$x + y - 3 = 0$$

which is

$$x - 3 = 0 .$$

The point of intersection is $(3,0,4)$. Similarly, the trace of the plane, $x + y - z + 1 = 0$, in the YZ-plane intersects the trace of $x + y - 3 = 0$ in the YZ-plane in the point $(0,3,4)$. These points are both in the plane $z = 4$. Thus, the line joining these 2 points is the line of intersection of the three planes.



Exercises 11. Miscellaneous Exercises - Answers.

1. The number is 364 .
2. $3x + 4y + 5z = a$,

   $4x + 5y + 6z = b$,

   $5x + 6y + 7z = c$.

   Eliminate x: $3x + 4y + 5z = a$,

   $$y + z = 4a - 3b ,$$

   $$2y + 2z = 5a - 3c .$$

   Condition: $5a - 3c = 2(4a - 3b)$

   or $a + c = 2b$
3. The number is 456 or 654 .
4. \$6500, \$1300, \$2200.
5. $a = 7$; the line is given by $\begin{cases} x = -2y + 7 \\ y \quad \text{arbitrary} \\ z = y - 1 \end{cases}$
6. 5 cu. yds., 6 cu. yds., 8 cu. yds.
7. 12 dimes, 8 nickels, 20 pennies.
8. 75 units, 80 units, 50 units.
9. 12 days, 8 days, 6 days.
10. 8 hours, 4 hours, 8 hours.

11. $AS = AR = 4\frac{1}{2}$; $BS = BT = 5\frac{1}{2}$; $CT = CR = 2\frac{1}{2}$

12. $y = -x^2 + 2x + 4$

13. $y = 3x^2 + 2x - 1$

14. 160 elementary school pupils
    80 high school pupils
    80 adults

15. $K = -8$; $A = 50$, $B = 0$

16. Rewrite the given equation

$$w_1T + w_2Q + w_3E = (w_1 + w_2 + w_3)A$$

as

$$w_1(T - A) + w_2(Q - A) + w_3(E - A) = 0.$$

Using the table of scores we construct the following table:

	T - A	Q - A	E - A
Frank	-4	-4	4
Joyce	-2	18	-6
Eunice	3	-17	5

from which we write our system

$$w_1 + w_2 - w_3 = 0 ,$$
$$w_1 - 9w_2 + 3w_3 = 0 ,$$
$$3w_1 - 17w_2 + 5w_3 = 0 .$$

Eliminate x:

$$w_1 + w_2 - w_3 = 0 ,$$
$$-10w_2 + 4w_3 = 0 ,$$
$$-20w_2 + 8w_3 = 0 .$$

Each of the last two equations reduces to

$$5w_2 - 2w_3 = 0.$$

So $w_2 = \frac{2}{5}w_3$ and $w_1 = w_3 - w_2 = \frac{3}{5}w_3$.

(Equivalently, $w_1 : w_2 : w_3 = 3:2:5$)

For $w_1 + w_2 + w_3 = 1$, we can write

$$\frac{3}{5}w_3 + \frac{2}{5}w_3 + w_3 = 1$$

or $10w_3 = 5$ so $w_3 = 0.5$, $w_2 = 0.2$, $w_1 = 0.3$.

17. Let a = number of air mail stamps purchased

f = number of 4 cent stamps purchased

s = number of one cent stamps purchased

$$.07a + .04f + .01s = 10$$
$$a = 2f$$

observe that there are only 2 equations in 3 unknowns. However, s must be an integer less than 18 (the price of one 4 cent stamp and 2 air mail stamps) since only the change is spent for 1 cent stamps.

Substituting $a = 2f$ in

$$7a + 4f + s = 1000$$

we have

$$18f + s = 1000$$
$$s = 1000 - 18f$$

$f = 55$ is the largest integral value that leaves s positive. Therefore

$$f = 55 \ , \ s = 10 \ , \ a = 110 .$$

Note that the problem can be solved simply by observing that we are to buy the largest number possible of 18 cent units consisting of 2 air mail and 1 four cent stamp), and spend the change on 1 cent stamps.

18. Actual score is 81. Reported score is 63. Par is 60.

*19. If A, B, C, D are the coefficients of our desired plane,

$$Ax + By + Cz + D = 0 ,$$

we obtain three equations for the four "unknowns" A, B, C, D by demanding that the coordinates of the three given points shall satisfy this equation:

$$A \cdot (-1) + D = 0 ,$$
$$A \cdot 1 + B \cdot (-1) + D = 0 ,$$
$$A \cdot (-1) + B \cdot 3 + C \cdot 2 + D = 0 ;$$

or

$$-A + D = 0 ,$$
$$A - B + D = 0 ,$$
$$-A + 3B + 2C + D = 0 .$$

Eliminating A from the second and third:

$$-A + D = 0 ,$$
$$-B + 2D = 0 ,$$
$$3B + 2C = 0 .$$

41

Eliminating B from the third:

$$-A + D = 0 ,$$
$$-B + 2D = 0 ,$$
$$2C + 6D = 0 .$$

Hence $A = D$, $B = 2D$, $C = -3D$,

an answer being $x + 2y - 3z + 1 = 0$ $(D = 1)$.

Since any other choice of D will give an equation with coefficient proportional to these, only one plane is determined.

20. A. (a) Ag + (b) $HNO_3 \longrightarrow$ (c) Ag NO_3 + (d) NO + (e) H_2O

Ag: a = c; H: b = 2e; N: b = c + d; O: 3b = 3c + d + e

$3Ag + 4HNO_3 \longrightarrow 3AgNO_3 + NO + 2H_2O$

B. (a) $AuCl_3$ + (b) KI $\longrightarrow$ (c) AuCl + (d) KCl + (e) I_2

Au: a = c; Cl: 3a = c + d; K: b = d; I: b = 2e

$AuCl_3 + 2KI \longrightarrow AuCl + 2KCl + I_2$

C. (a) HNO_3 + (b) HI $\longrightarrow$ (c) NO + (d) I_2 + (e) H_2O

H: a + b = 2e; N: a = c; O: 3a = c + e; I: b = 2d

$2HNO_3 + 6HI \longrightarrow 2NO + 3I_2 + 4H_2O$

D. (a) MnO_2 + (b) HCl $\longrightarrow$ (c) $MnCl_2$ + (d) Cl_2 + (e) H_2O

Mn: a = c; O: 2a = e; H: b = 2e; Cl: b = 2c + 2d

$MnO_2 + 4HCl \longrightarrow MnCl_2 + Cl_2 + 2H_2O$

E. (a) $Cr(OH)_3$ + (b) NaOH + (c) $H_2O_2 \longrightarrow$ (d) Na_2CrO_4 + (e) H_2O

Cr: a = d; O: 3a + b + 2c = 4d + e; H: 3a + b + 2c = 2e;

Na: b = 2d

$2Cr(OH)_3 + 4NaOH + 3H_2O_2 \longrightarrow 2Na_2CrO_4 + 8H_2O$

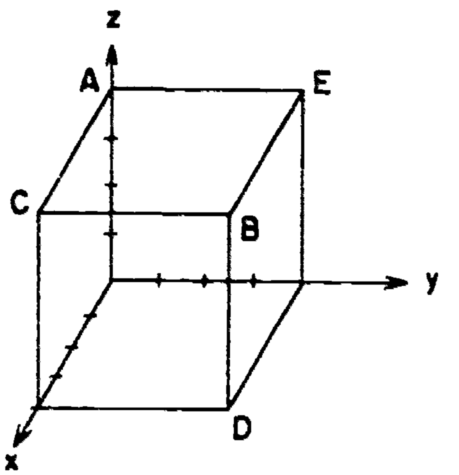
Illustrative Test Questions

Part I: Multiple Choice.

Directions: Select the response which best completes the statement or answers the question.

1. The set of points in space equidistant from two given points is
   (a) a cylinder.
   (b) a plane.
   (c) a straight line.
   (d) the midpoint of the line segment which joins the two points.
   (e) two parallel straight lines.

2. The point whose coordinates are (4,0,4) is
   (a) A.
   (b) B.
   (c) C.
   (d) D.
   (e) E.



3. Which of the following is an ordered triple of real numbers that corresponds to a point in the xz-plane?
   (a) (0,2,0).
   (b) (0,3,-2).
   (c) (3,2,0).
   (d) (2,3,2).
   (e) (-2,0,3).

4. The distance between the points (2,3,4) and (4,3,2) is
   (a) 0.
   (b) 4.
   (c) $2\sqrt{2}$.
   (d) 8.
   (e) $3\sqrt{2}$.

5. Which one of the following points is 5 units from the origin?
   (a) (-4,3,0).
   (b) (1,2,0).
   (c) $(\sqrt{2},1,\sqrt{2})$.
   (d) $(\sqrt{2},\sqrt{3},0)$.
   (e) (5,3,4).

6. The equation $ax + by + cz + d = 0$, where a, b, c, d are real constants, represents a plane if and only if

(a) all four constants are different from zero.

(b) $d \neq 0$

(c) a, b, c are all different from zero.

(d) at least one of the constants a, b, c, is different from zero.

(e) at least one of the constants a, b, c, d, is different from zero.

7. Which of the following statements about the plane whose equation is $x + y + z = 0$ is <u>not</u> true?

(a) It is the perpendicular bisector of the line segment joining. $(1,1,1)$ and $(-1,-1,-1)$.

(b) It passes through the origin.

(c) It contains the point $(0,1,-1)$.

(d) It intersects the xy-plane in the line $x + y = 0$.

(e) It intersects the z-axis in the point $(1,-1,0)$.

8. The set of points in space defined by the equation $y = 5$ is

(a) a plane parallel to the y-axis.

(b) a plane perpendicular to the y-axis.

(c) a plane containing the y-axis.

(d) a line intersecting the y-axis.

(e) a point on the y-axis.

9. What is the equation of the plane whose graph is sketched at the right?

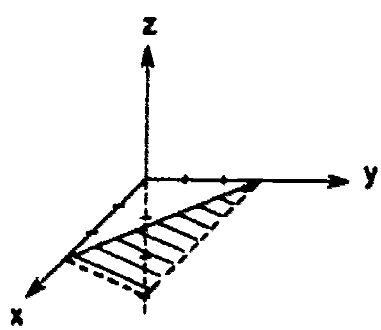
(a) $x + y = 3$.

(b) $x + y - z = 3$.

(c) $-x - y + z = 3$.

(d) $x - y + z = 3$.

(e) $x - y - z = 3$.



10. Which one of the following points lies in the plane whose equation is $x - 2y = 6$?

(a) $(0,-3,9)$.

(b) $(2,2,7)$.

(c) $(0,6,0)$.

(d) $(0,3,-6)$.

(e) $(12,-3,6)$.

11. The solution set of the equation $px + qy + rz = 0$ contains the element

(a) $(p,q,-r)$.

(b) $(r,-p,q)$.

(c) $(0,0,r)$.

(d) $(0,r,q)$.

(e) $(r,0,-p)$.

12. Which of the following number triples is in the solution set of the system $\begin{cases} x - 2y + z = 4 \\ z = 2 \end{cases}$?

I. $(2,0,2)$.

II. $(0,-2,0)$.

III. $(4,1,2)$.

(a) I. only

(b) II. only

(c) III. only

(d) I. and III. only

(e) I., II. and III

13. How many number triples are in the solution set of three equations which represent three coincident planes?

(a) 0

(b) 1

(c) 2

(d) 3

(e) Infinitely many

14. The trace of the graph of the equation $x - 2y + z = 5$ in the xy-plane is

(a) $-2y + z = 5$.

(b) $x - 2y = 5$.

(c) $x + z = 5$.

(d) $x - 2y = 0$.

(e) $x + z = 0$.

15. The trace of the graph of the equation $ax + by + cz = d$ in the xz-plane is given by

(a) $by = d$.

(b) $\begin{cases} ax + by + cz = d \\ y = 0 \end{cases}$

(c) $ax + cz = 0$.

(d) $x + z = \dfrac{d}{a + c}$.

(e) none of the above.

16. Which of the following represents a straight line in a three-dimensional coordinate system?

(a) $\begin{cases} x = -3z + 1 \\ y = 2z + 3 \end{cases}$

(b) $\begin{cases} x + y = 6 \\ x + y = 7 \end{cases}$

(c) $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$

(d) $x = y$

(e) $x = 3$

17. In each of the following systems the three equations represent three planes. In which system do the three planes intersect in a line?

(a) $\begin{cases} z = 0 \\ y = 0 \\ z + y = 2 \end{cases}$

(b) $\begin{cases} x = 2 \\ y = 4 \\ 2x - y = 0 \end{cases}$

(c) $\begin{cases} x = 0 \\ y = 0 \\ z = 0 \end{cases}$

(d) $\begin{cases} x - 2y = 0 \\ x - 2y = 4 \\ z = 5 \end{cases}$

(e) $\begin{cases} x + y + z = 4 \\ 2x + 2y + 2z = 8 \\ 3x + 3y + 3z = 12 \end{cases}$

18. Which statement is true of the solution set of the following system of equations?

$$\begin{cases} 3x - y + 2z = 6 \\ 6x - 2y + 4z = 7 \end{cases}$$

(a) The solution set has an infinite number of elements.
(b) The graph of the solution set is a straight line.
(c) The solution set is empty.
(d) The solution set contains exactly one element.
(e) None of the above statements is true.

19. Which one of the following systems of equations represents a pair of parallel planes?

(a) $\begin{cases} 2x + 3y + 4z = 0 \\ x + 2y + 3z = 0 \end{cases}$

(b) $\begin{cases} 2x + 3y - 4z = 1 \\ 2x + 3y + 4z = 1 \end{cases}$

(c) $\begin{cases} 2x - 3y - 4z = 3 \\ 2x - 6y - 8z = 6 \end{cases}$

(d) $\begin{cases} 2x + 3y - 4z = 2 \\ 4x + 6y - 8z = 2 \end{cases}$

(e) $\begin{cases} x = 4 \\ y = 3 \end{cases}$

20. The solution set of the system $\begin{cases} x + y + z = 2 \\ 3x - 3y + 3z = 9 \\ x + y - z = 6 \end{cases}$

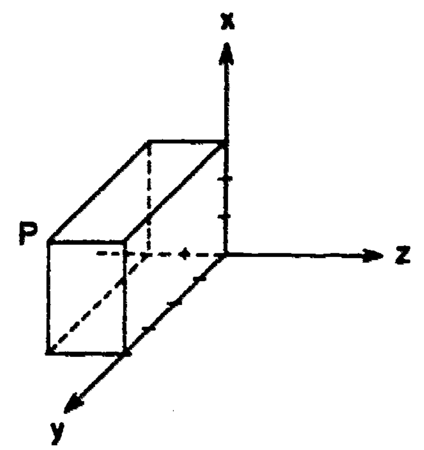
(a) is empty.
(b) contains a single number triple.
(c) contains an infinite set of number triples which correspond to points of a straight line.
(d) contains an infinite set of number triples which correspond to points of a plane.
(e) contains exactly three number triples.

21. What is the solution set of the following system? $\begin{cases} x + y + z = 4 \\ x + y = 2 \\ y = -3 \end{cases}$

(a) (-1,3,2).
(b) (1,-3,6).
(c) (5,-3,2).
(d) (5,-3,-4).
(e) (-5,-3,12).

Part II: Problems.

22. If the x, y, and z axes are chosen as shown in the figure, what triple of real numbers (x,y,z) are the coordinates of P ?



23. Find the distance between the points (3,4,2) and (-3,4,0).

24. Find an equation for the locus of points equidistant from the points (2,4,-1) and (0,5,6).

25. Make a free-hand drawing of the graphs of the following equations in a three-dimensional coordinate system.

(a) $x + y = 2$.
(b) $3x + y + 2z = 6$.

26. If the planes whose equations are given in the following system intersect in a line, express two of the variables of the solution set in terms of the third variable. If the planes do not intersect in a line, describe their position with respect to each other.

$$\begin{cases} x + 3y - 2z = 6 \\ x - 2y + z = 4 \end{cases}$$

27. Find the solution set of the following system of equations.

$$\begin{cases} 2x - 4y + 3z = 17 \\ x + 2y - z = 0 \\ 4x - y - z = 6 \end{cases}$$

28. Find a three-digit number such that the sum of the digits is 19; the sum of the hundreds digit and the units digit is one more than the tens digit, and the hundreds digit is four more than the units digit.

29. Three tractors, A, B, and C, working together can plow a field in 8 days. Tractors A and B can do the work in 14 days. Tractor A can plow the entire field in half the time that it takes Tractor C. Write a system of equations which could be solved to find the number of days it would take each tractor to do the work alone.
(You need not solve the system).

30. Give the coordinates of the point which is symmetric to the point $(1,-2,3)$ with respect to
(a) the origin.
(b) the x-axis.
(c) the y-axis.
(d) the z-axis.
(e) the yz-plane.
(f) the zx-plane.
(g) the xy-plane.

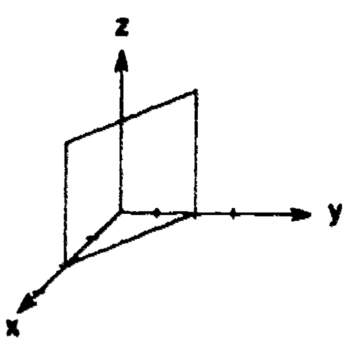
Answers to Illustrative Test Questions

Part I. Multiple Choice:

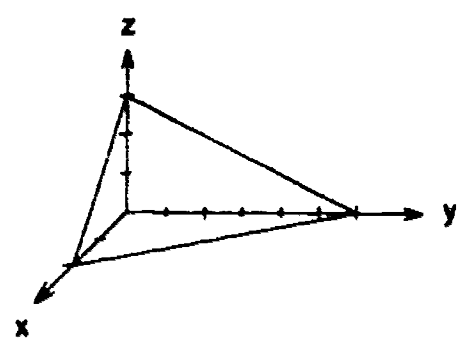
1. B
2. C
3. E
4. C
5. A
6. D
7. E
8. B
9. B
10. A
11. E
12. D
13. E
14. B
15. B
16. A
17. B
18. C
19. D
20. B
21. C

Part II. Problems:

22. $(3,4,-2)$
23. $2\sqrt{10}$
24. $2x - y - 7z + 15 = 0$
25. (a)



(b)



26. $\begin{cases} x = \dfrac{z + 24}{5} \\ y = \dfrac{3z + 2}{5} \end{cases}$ or $\begin{cases} y = 3x - 14 \\ z = 5x - 24 \end{cases}$ or $\begin{cases} x = \dfrac{y + 14}{3} \\ z = \dfrac{5y - 2}{3} \end{cases}$

27. (3,1,5)

28. 793

29. $\begin{cases} \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{C} = \dfrac{1}{8} \\ \dfrac{1}{A} + \dfrac{1}{B} = \dfrac{1}{14} \\ \dfrac{1}{A} = \dfrac{2}{C} \end{cases}$

30. (a) (-1,2,-3)
(b) (1,2,-3)
(c) (-1,-2,-3)
(d) (-1,2,3)
(e) (-1,-2,3)
(f) (1,2,3)
(g) (1,-2,-3)